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Abstract

The work to compute the third precise levelling in Sweden has mainly been performed as a Nordic cooperation under the umbrella of the Nordic Geodetic Commission (NKG) within the Working Group for Height Determination. It includes the compilation of the Baltic Levelling Ring, consisting of precise levellings from all the Nordic and Baltic countries as well as Poland, Germany and the Netherlands. Due to the acute need of a new system, Sweden had to finalise the project at the beginning of 2005. It was decided that the Swedish height system (frame) RH 2000 should be a realisation of the European Vertical Reference System (EVRS) using the Normaal Amsterdams Peil (NAP) as zero level. Presupposing these choices, the most crucial part of the definition of RH 2000 is the specification of a model for the reduction of postglacial rebound.

The main purpose of this report is to discuss the system definition and to present the work to construct a suitable land uplift model for the RH 2000 adjustment of the Baltic Levelling Ring. The path leading to the model is treated in great detail. The final uplift model is a combination of the geophysical model of Lambeck, Smither and Ekman with the mathematical model of Vestøl. We also analyse the consequences of the chosen definition and land uplift model by comparing the resulting heights to Mean Sea Level in the Nordic and Baltic Seas and to a few other height systems.

The land uplift model was adopted as a Nordic model by the NKG in 2006 and was then renamed from RH 2000 LU to NKG2005LU. The RH 2000 adjustment of the BLR has also been accepted as giving the final result of the BLR project.

Acknowledgments

We are in great depth to *all* members of the NKG Working Group for Height Determination, which have contributed extensively to this project. Specifically we can mention Jaakko Mäkinen, Olav Vestøl, and Karsten Engsager. Without this Nordic co-operation it would not have been possible for Sweden to finish the computation of the third precise levelling.

We further thank Johannes Ihde and Martina Sacher, EUREF (IAG Subcommission for Europe), and the responsible agencies for providing the levelling data of the non-Nordic countries to the Baltic Levelling Ring project.

We are especially indebted to Martin Ekman for an uncountable number of discussions, as well as for reading and commenting on an early version of this document. His work on land uplift and on other geodynamic phenomena affecting levelling has been immensely valuable to this work.

1. Introduction

1.1 The postglacial rebound of Fennoscandia

The postglacial uplift of Fennoscandia has been extensively studied during the last two hundred years; see Ekman (1991) for a historical review. Until recently, the most important way to determine the vertical uplift has been to utilise sea and lake level observations together with repeated precise levellings. For instance, the model presented by Ekman (1996) was derived using high quality sea level observations from 58 tide gauges in the Baltic and surrounding seas, lake level observations and repeated precise levellings from the Nordic countries. The postglacial land uplift can also be determined using GPS and other space geodetic techniques. One notable example in this direction is provided by the BIFROST project (e.g. Johansson et al. 2002), in which the rebound is observed at approximately 50 permanent GPS stations that cover more or less the whole of the Fennoscandian area. In the beginning (the project in question started 1993), the uplift rates from GPS were not as accurate as the tide gauge counterparts, but the situation naturally improves as time passes. In addition, most of the hardware problems, which degraded the quality in the earlier years, have been satisfactorily solved. Today, almost 10 years of continuous GPS observations are available and the accuracy improves constantly; see the latest uplift rates from the BIFROST project (Lidberg 2004). It should also be mentioned that other ways to determine the uplift exist, for instance to study ancient shore lines.

No matter what land uplift observations that are utilised, basically two different ways exist to derive a continuous model from the discrete observations. The first option is to view the construction of a land uplift model as a pure interpolation (possibly extrapolation) problem. We have a set of observations with different geographic locations and quality, from which the best possible continuous surface is to be constructed using a suitable mathematical technique. In this report, a model of this type will be called a *mathematical model*. Danielsen (1998) developed a technique to determine the land uplift from “non-repeated” precise levellings, in which each line is observed only once, at the same time as different lines are observed at different epochs. The mathematical method used there is least squares collocation with unknown parameters (Moritz 1980). This method was then refined and applied by Vestøl (2002, 2005), which finally included almost all available Nordic GPS, levelling and tide gauge observations for the construction of a land uplift model (Vestøl 2005).

The second way to construct an uplift model is to make use of physical theories for how the Earth responds to the melting at the end of the last ice age. A model of this type will be called a *geophysical model* below. A number of geophysical models have been proposed during the years; see Ekman (1991) for a historical review. The latest ones are extremely complicated: The fewer assumptions concerning the Earth's physical constitution that are used, the more complex the model becomes. One relevant example here is provided by the model of Lambeck et al. (1998), which was constructed to fit tide gauge and shore line observations. An elastic lithosphere of a certain thickness with a comparatively high rigidity is taken to be situated over a two-layer mantle, which is assumed to behave as a viscous fluid for the time scales relevant for postglacial rebound. A model for the ice sheet is also devised. The geophysical model is tuned to the tide gauge observations by varying the lithosphere thickness, the viscosities of the two mantle layers and by modifying the ice model. Similar models have also been constructed in the BIFROST project (applying, however, the Lambeck ice model), but here GPS velocities have been used for tuning; see Milne et al. (2004). It should be noticed that a geophysical model makes it possible to take advantage of other knowledge than direct observations of the uplift. For instance, from the fact that the lithosphere is known to behave in a comparatively rigid way (elastic with high flexural rigidity), it follows that the uplift rate cannot vary arbitrarily, i.e. a smooth velocity field is implied. On the other hand, we will not accept physical parameters that disagree with our observations, considering of course the accuracy of these. In the context of constructing the best possible land uplift model, the use of a geophysical method may be viewed as a complicated interpolation (and extrapolation) scheme, where the interpolation is controlled by the physical parameters of the Earth (including the ice). Whether this interpolation is actually correct, is of course determined by how realistic the model is.

1.2 The Baltic Levelling Ring

The processing of the latest precise levellings of Sweden, Finland and Norway has been made as a Nordic co-operation under the auspices of NKG. Denmark also contributed actively to the task, even though the Danish height system DVR 90 had already been finalised (Schmidt 2000). To be able to connect to the Normaal Amsterdams Peil (NAP), which is the traditional zero level for the United European Levelling Network (UELN), and to be able to determine the relations to our neighbouring countries, it was decided to extend the Nordic network with the precise levellings from the Baltic States, Poland, Northern Germany and the Netherlands. The non-Nordic data was provided by EUREF from the UELN-database.

The whole network, which has been named the Baltic Levelling Ring (BLR), is illustrated in Fig. 1.1. Unfortunately, it has not been possible to close the ring with levelling observations around the Gulf of Finland. However, by means of other information (sea surface topography or GPS in combination with a geoid model), closing errors may still be computed. This amounts to a valuable check of the adjustment. It should be noticed, though, that only levelling observations are included in the final adjustment.

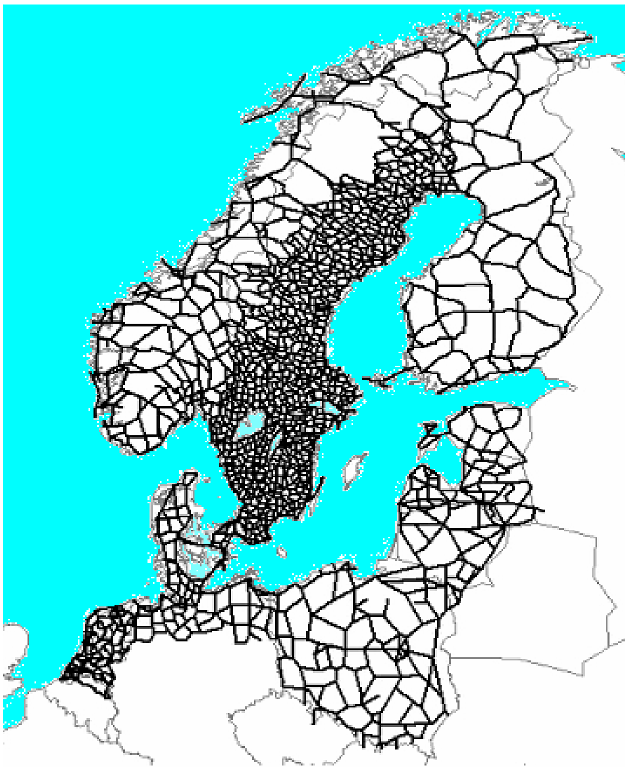

Figure 1.1 The Baltic Levelling Ring (BLR) network.

1.3 Choice of system definition for RH 2000

Now, due to the phenomenon of land uplift, it is crucial in the Nordic area to reduce all levelling observations to a common reference epoch. It might even be argued that the specification of uplift model constitutes the most important part of the *system definition* for a national height system in the Nordic countries. When the new height system RH 2000 was to be defined for the computation of the third precise levelling in Sweden, mainly the following key choices were discussed in collaboration with the other Nordic countries under the umbrella of the Nordic Geodetic Commission (NKG):

- Land uplift model (mathematical, geophysical or a combination).
- Reference epoch (middle of the observations, i.e. 1990, or 2000.0)
- Zero level (NAP or wait for a World Height System)
- Type of heights (normal or some type of orthometric)
- Permanent tide system (zero, non-tidal or mean)

These discussions have been documented in a long row of publications; see for instance Mäkinen et al. (2004, 2005). In order to arrive at European height systems agreeing well with each other, it might seem suitable that the national systems *should* be defined according to the definitions of the Technical Working Group of the IAG Subcommission for Europe (EUREF); cf. Ihde and Augath (2001). One problem here, though, is that the 2005 definition of the European Vertical Reference System (EVRS) is very general; it includes almost any height system using normal heights together with a zero permanent tide. This gives each country a considerable freedom concerning how their national system should be defined. One way to realise EVRS was taken in the computation of the United European Levelling Network 95/98 (UELN 95/98), which resulted in the European Vertical Reference Frame (EVRF 2000). This realisation was made using the Normaal Amsterdams Peil (NAP) as zero level in the traditional European way.

The system definition discussions within the NKG have been quite general (e.g. Mäkinen et al. 2004; Mäkinen 2004). It has for instance been questioned whether NAP is the most suitable way to fix the zero level. Is it not better to wait for a so-called World Height System

(WHS), which is fixed using GPS and a global geoid model of cm-accuracy? From the Swedish perspective, it has not been possible to wait for such developments. Due to the extremely high requirements on the geoid model, it might also be questioned whether it will really become possible to determine a World Height System with sufficient accuracy in the foreseeable future. In any case, no World Height System will be available soon enough. For the final computation of the third precise levelling in Sweden, it was therefore decided to follow the then European recommendations available in 2005 as far as possible. This means that the resulting system (RH 2000) becomes a realisation of the European Vertical Reference System (EVRS), which is made according to similar principles as applied for the computation of the European Vertical Reference Frame (EVRF 2000). Consequently, it is already specified that the Normaal Amsterdams Peil (NAP) is used to define the zero level, that normal heights are utilised and that the system is of a zero permanent tide type. However, no EUREF recommendation was available in 2005 concerning how the land uplift should be taken care of in the Nordic area nor of which reference epoch that should be utilised. In fact, in the computation of EVRF 2000, the levelling observations were not even reduced to a common epoch. This means that the Nordic Block in EVRF 2000 has the land uplift epoch 1960.0, to which the Swedish, Finnish and Norwegian observations were reduced before delivery to the UELN computing centre in 1980 (Mäkinen et al. 2004).

It remains to choose a suitable land uplift model and a reference epoch to which all levelling observations are to be reduced. Concerning the epoch, it is naturally most optimal with the mean of all observations, since this will minimise the influence of errors in the uplift model; see for instance Ekman (1995). This question was decided in cooperation with the other Nordic countries. Now, due to political reasons, Finland did not consider it possible to use an epoch in the 1990ties. The reference epoch was therefore chosen to 2000.0, which is a reasonable compromise not too far removed from the mean of the observations, but sufficiently correct from a political point of view.

1.4 Purpose and content

The last and most important part of the system definition is how the land uplift model is chosen. It is the *main purpose* of this report to present the land uplift model that is used in the RH 2000 adjustment of the Baltic Levelling Ring, which resulted in the new Swedish height system RH 2000. This task includes a detailed presentation of the relevant background and the work behind the model performed at Lantmäteriet (National Land Survey of Sweden) in cooperation with the Working group for height determination within the Nordic Geodetic Commission (NKG). It should be noticed that due to severe time limitations, it was neither possible to wait for the perfect model to emerge on the market (so to speak) nor to investigate all possible ways to construct new models from scratch. Instead it was decided to start from two already existing ones, namely the mathematical model of Vestøl (2005) and the geophysical counterpart of Lambeck et al. (1998), and to combine or modify them in such ways that the final model fulfils the present purpose sufficiently well. This means that the criteria for choosing the final model depend on how this affects the adjusted heights in RH 2000. If two models give almost exactly the same heights, they are considered as equally good. Notice, however, that this does not necessarily mean that the two models are equally good for all tasks.

One special requirement on the uplift model stems from the fact that the adjustment of RH 2000 is made using levelling observations for the whole *Baltic Levelling Ring network* illustrated in Fig. 1.1, which includes observations from all countries around the Baltic Sea. This means that the land uplift model must cover a very large area. Unfortunately, the observations do not extend sufficiently far to make it possible to take advantage of Vestøl's model as it is. A good way to extend Vestøl's model, however, seems to be to make use of Lambeck's geophysical model outside the area where Vestøl's model is defined. This path was also chosen by the NKG height determination group. One such composite model was thus constructed by Karsten Engsager (NKG height determination working group email) in Denmark using least squares collocation. However, it is believed that it is far from evident how the two models should be combined. One specific purpose of this report is therefore to investigate a few different methods to extend Vestøl's model outside its definition area using Lambeck's geophysical model.

Another alternative that was seriously considered within NKG was to utilise only Lambeck's model. As mentioned above, this model is tuned to the tide gauges within the Nordic area. It should thus be good along the coasts. In areas without tide gauges, however, the quality is more questionable. One advantage with using Lambeck's model is that this one is geophysical. This means that it represents a reasonably realistic representation of the land uplift field, which takes advantage of other types of knowledge to make a realistic smoothing, interpolation and extrapolation of the tide gauge observations. It is another aim of this report to investigate the merits of Lambeck's model by itself and to compare it to Vestøl's counterpart.

It is obviously important that the land uplift model is realistic: It should represent the uplift with as little observation errors as possible. As mentioned above, one way to obtain a realistic field is to use a geophysical model, but as no available geophysical model takes all the available data into account, it was finally decided to choose a mathematical model for RH 2000. One problem with this is that it is difficult to select the relevant parameters. As argued above, it follows from the high flexural rigidity of the lithosphere that the velocity field should be reasonably smooth. The mathematical model of Vestøl (2005), which is our starting point, is already smoothed to a certain degree (see Chapter 2). Another purpose of the present report is to investigate the question of *smoothing* a little further, and to find out how the amount of smoothing affects the adjusted RH 2000 heights. Of course, there is no way to escape the observation errors in order to reach the "true" uplift, but it is nevertheless believed that it is important to take this question seriously. The tuning of the model by the choice of covariance function and apriori standard errors corresponds to the choice of physical Earth (and ice) parameters for a geophysical model. The strategy here is that the mathematical model should "look" realistic at the same time as it should fit the given observations as well as possible. It should be noticed, though, that the fit to the observations cannot be the only criterion for the construction of a mathematical model. It is always possible to choose a very rough model that fits all observations perfectly. Needless to say, such a model is useless for the present task. It would leave the door wide-open for *old* levelling errors to affect the *new* height system.

As discussed above, the choice of uplift model is a very important part of the definition of RH 2000. The other four parameters in the above list were either decided on a European or a Nordic level. It is important to notice, though, that the specification of land uplift model is not totally separated from the choice of zero level. Since the NAP is affected by the land uplift phenomenon (it sinks), it might be thought that the reference level itself should be corrected for the uplift. This, however, does not fit with the way NAP has been treated on the European level (EVRF 2000). Another purpose of the present report is to carefully delineate the 2005 definition of the European vertical reference system as well as of RH 2000, and to investigate the consequences of the land uplift in Amsterdam (NAP). It is further the aim is to investigate the final product of the chosen system definition, i.e. the adjusted heights in RH 2000, which includes comparisons with the old Swedish height system RH 70, with EVRF 2000 and with the new Danish system DVR 90 (Schmidt 2000). Another consequence is that the definition determines the height of the Mean Sea Level (MSL). It is finally also the purpose to study the MSL for a few tide gauges along the Swedish coast.

The report has been organised in the following way. The basic uplift observations are introduced in Chapter 2, which also presents and analyses the Vestøl and Lambeck models. Chapter 3 then treats the work performed at Lantmäteriet to find a suitable land uplift model for the RH 2000 computation of the Baltic Levelling Ring. This includes work starting not only from Vestøl's model in gridded form, but also from the estimated uplift values in the observation points themselves. It is constantly assumed that Vestøl's model is extended with Lambeck's counterpart. Chapter 3 also contains investigations of different interpolation methods and the degree of smoothing. It ends with a small study of the way the interpolation schemes affect the closing errors around the Gulf of Bothnia and the Baltic Sea. In Chapter 4, the definitions used on the European level (EVRS and EVRF 2000) in 2005 are first described in more detail compared to above, which is followed by a discussion of the definition of RH 2000. In connection with this, the consequences of the land sinking at the NAP are discussed and investigated numerically. After the final model has been chosen, it is evaluated by a detailed comparison with the observations. The adjusted RH 2000 heights are also compared with those of the old Swedish height system RH 70 and with EVRF 2000. A small investigation of the height of the Mean Sea Level (MSL) along the Swedish coast is also presented. The report ends with a general discussion and summary.

1.5 Note added in 2007

This report was written in its entirety during 2005, but is not published until now (2007). In the original version, the RH 2000 Land Uplift model was called RH 2000 LU. Since then the land uplift model has been adopted as a Nordic model by the NKG and has received the new name NKG2005LU. The RH 2000 adjustment has also been accepted as giving the final solution of the BLR project. In order to avoid a complete rewriting, the report is kept in its original shape. The only exceptions are:

- RH 2000 LU is renamed NKG2005LU throughout the report.
- The addition of this and a similar one at the end of the report, which explains the development since 2005.
- The year 2005 is added to some statements to indicate that they refer to the situation that year, for instance the 2005 version of the European Vertical Reference System (EVRS).

2. Vestøl's and Lambeck's uplift models

The main purpose of this chapter is to present and analyse the land uplift models presented by Vestøl (2005) and Lambeck et al. (1998), which are the starting points for the present work. As the land uplift observations used by Vestøl (ibid.) will be applied also to evaluate Lambeck's model, the chapter starts with giving a short account of the observations. After that, Vestøl's and Lambeck's models are treated in turn.

2.1 Available observations

The basic observations applied by Vestøl are the apparent land uplift rates at 58 tide gauges published by Ekman (1996), 55 absolute GPS velocities from the BIFROST project (Lidberg 2004) and precise levelling observations from Sweden, Finland and Norway.

The apparent uplift rates at the 58 *tide gauges* were computed using linear regression by Ekman (1996). All observations were reduced to the common 100 years period 1892–1991 in order to eliminate oceanographic changes. This interval was chosen so that extreme high and low water years are avoided at the beginning and end of the period. To correct those sea level series that do not cover the whole period, two reference stations were used, one in the Baltic Sea (Stockholm) and one in the North Sea (Smögen). The resulting apparent uplift values are summarised in Fig. 2.1. The reader is referred to Ekman (1996, Table 1) for more details. As can be seen, the spatial distribution of the tide gauges is dense in the Baltic Sea and its transition into the North Sea, while it is less dense in the Norwegian and Arctic Seas. Only two mareographs are situated north of Trondheim in the latter case. The standard error for the uplift values is estimated by Ekman (1996) to 0.2 mm/year, even though the formal standard errors might be considerably smaller for the differential uplift between neighbouring tide gauges. Ekman argues that various instrumental problems and long term oceanographic effects make it necessary to use a more pessimistic figure. In what follows, 0.2 mm/year is assumed representative for the standard errors of the tide gauge observations. It should finally be pointed out that the observations in Furuögrund (8.8 mm/year) and Oslo (4.1 mm/year) have been marked as outliers. These two observations were excluded by Vestøl (2005) based on a detailed statistical analysis using all observations; see further the discussion in Subsection 2.2.2. Another feature that has been included in Fig. 2.1 is a dividing line that is applied in Chapter 3 to neglect the southernmost observations. The reader is advised to neglect this line for the time being. It is needed in Chapter 3.

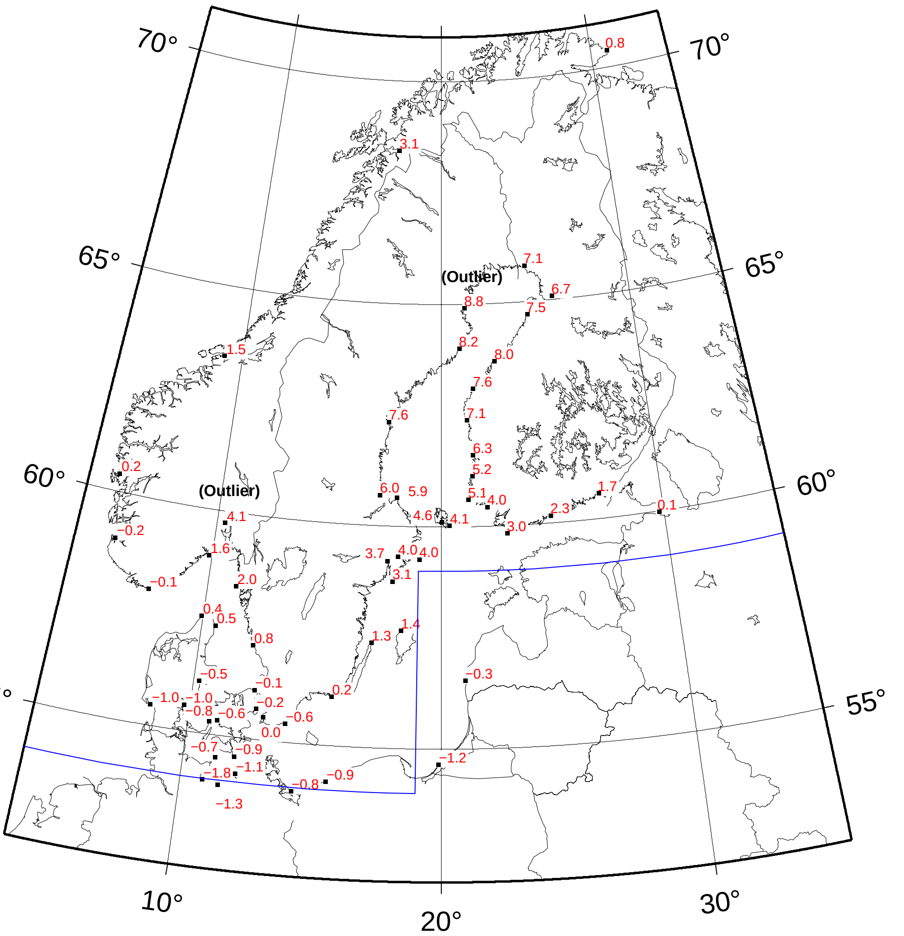

Figure 2.1: Apparent land uplift values at the *tide gauges* (Ekman 1996). Unit: mm/year.

The next group of uplift observations comprises the vertical GPS velocities from the latest BIFROST solution, which were estimated by Lidberg (2004) at 55 permanent GPS stations quite evenly distributed over the uplift area. A summary of the absolute uplift values can be found in Fig. 2.2; see Tables 1 and 4 in Paper D, Lidberg (2004) for details. The GPS uplifts stem from a systematic recomputation of approximately 3000 days (covering almost 10 years) of GPS observations using the GAMIT/GLOBK software. Two characteristic features of this solution are that an elevation cut off angle of 10 degrees is used and that the ambiguities are fixed to integers, contrary to earlier BIFROST solutions, which were computed with 15 degrees cut off as float solutions in GIPSY-OASIS (Johansson et al. 2002). It should further be mentioned that only very few changes have been made at the GPS stations since 1998, which means that no major hardware jumps occur after this year. The fact that the whole time series was recomputed in a unified way can also be expected to reduce the presence of systematic effects. A remaining problem, however, is the accumulation of snow on top of the radomes, which has not yet been satisfactorily solved. During the winter period, a large number of observations are therefore excluded as outliers, particularly in the northern parts of Sweden.

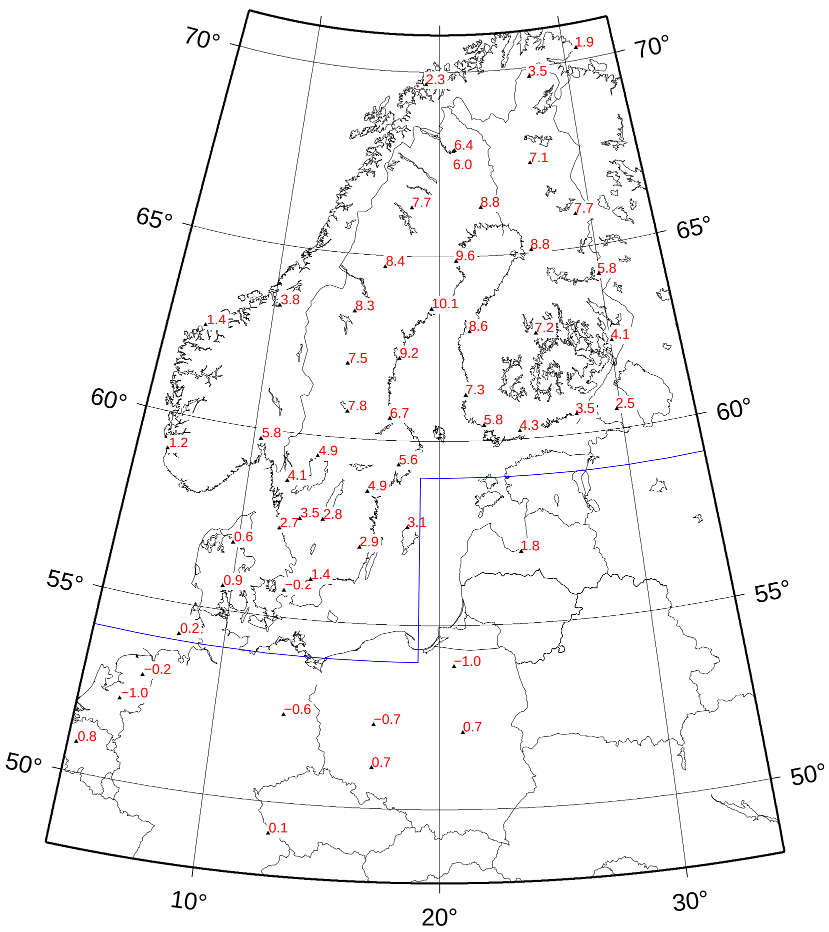

Figure 2.2: Absolute land uplift values at the *GPS-stations* (Lidberg 2004). Unit: mm/year.

Lidberg (2004) estimates standard errors for the velocities by first making linear regression of the edited time series. As these accuracy estimates assume zero correlation between the original observations (white noise distribution), the estimated standard errors are rescaled by a factor of 2 to 5 to take into account the influence of correlations; see Lidberg (2004) for further details. The resulting standard errors range from approximately 0.2 mm/year for the stations with the longest series, exemplified by a typical SWEPOS station, to 0.6 mm/year at some Norwegian and continental European stations. Lidberg then compares the estimated GPS velocities with absolute uplift values computed according to Ekman (1996) and Ekman and Mäkinen (1996a), and concludes that the real standard errors are a little higher than the rescaled formal counterparts. It might be considered realistic to assume a typical standard error of 0.3 – 0.4 mm/year for a good SWEPOS station, and perhaps the double of that for the more questionable stations in the central parts of Europe.

The final type of land uplift observation is the *precise levelling data* from Sweden, Finland and Norway. In both Sweden and Finland, three precise levellings have been performed. No systematic repeated levellings have been performed in Norway, but the observations nevertheless contain information on the land uplift, which is possible to estimate in case the rebound is modelled by some kind of continuous surface function, using for instance least squares collocation or a polynomial of suitable degree.

The Norwegian and Finnish precise levellings will not be considered in detail in the present report. Instead we concentrate on the Swedish situation. The epochs and standard errors in the three Swedish precise levellings are summarised in Table 2.1, while the lines are illustrated in Fig. 2.3. As can be seen, the network of the 3rd levelling is extraordinarily dense and homogeneous, but this is not the case for the 1st and 2nd counterparts. It should be noticed that for large parts of Sweden, only the 2nd and 3rd precise levellings exist, which are separated in time by 30 years on an average. The real time differences range from 12 years in the south to 48 years in the northern parts of Sweden; see Fig. 2.4. To get a feeling for what accuracy that can be expected, a simple error propagation was made, assuming that the levellings are separated by 30 years and using the standard errors in Table 2.1. In this case the relative uplift difference can be determined with the standard error 0.063 $\mathrm{mm}/\left(\mathrm{year}\cdot\sqrt{\mathrm{km}}\right)$, which implies 0.45 mm/year for 50 km, 0.63 mm/year for 100 km and 0.89 mm/year for 200 km distance. Thus, assuming that the levelling errors are random, it is not possible to do better than approximately 0.5 mm/year (1 sigma) for those large areas covered by only the 2nd and 3rd levellings. Of course, if systematic and gross errors are present, the errors are likely to increase even more. The most crucial problem with the second levelling is the low reliability, which implies that it is likely that a number of gross errors have not been detected and removed. The situation improves for the limited areas where all three levellings are available (see Fig. 2.3), but the fact that the quality of the 1st levelling is questionable (see Table 2.1), limits the accuracy also in this case. It should be pointed out that the above error propagation is made using the standard errors for unadjusted levelling lines. It is admitted that it would have been more correct to propagate the estimated standard errors for the *adjusted* height differences. However, due to the low redundancy of the first and second precise levellings, it is believed that the above results are fairly reasonable.

Table 2.1: Some information on the repeated precise levellings in Sweden. The information on the 1st and 2nd levellings was taken from Ekman (1996).

Levelling	Time	Mean Epoch	$\hat{s}_0$ [mm/$\sqrt{\text{km}}$]	System
1st	1886 – 1905	1892	4.4	RH 00
2nd	1951 – 1967	1960	1.6	RH 70
3rd	1979 – 2003	1990	1.0	RH 2000

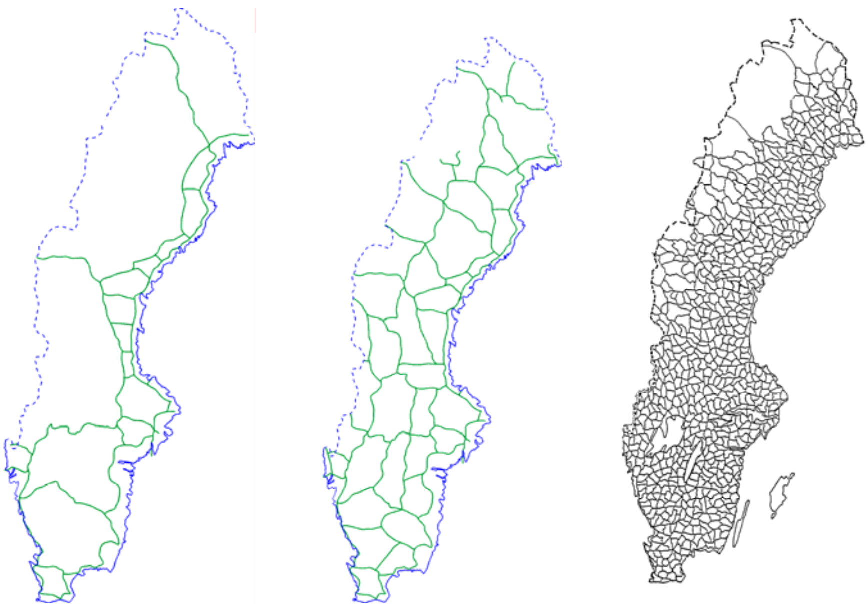

Figure 2.3: The levelling lines of the three precise levellings in Sweden.

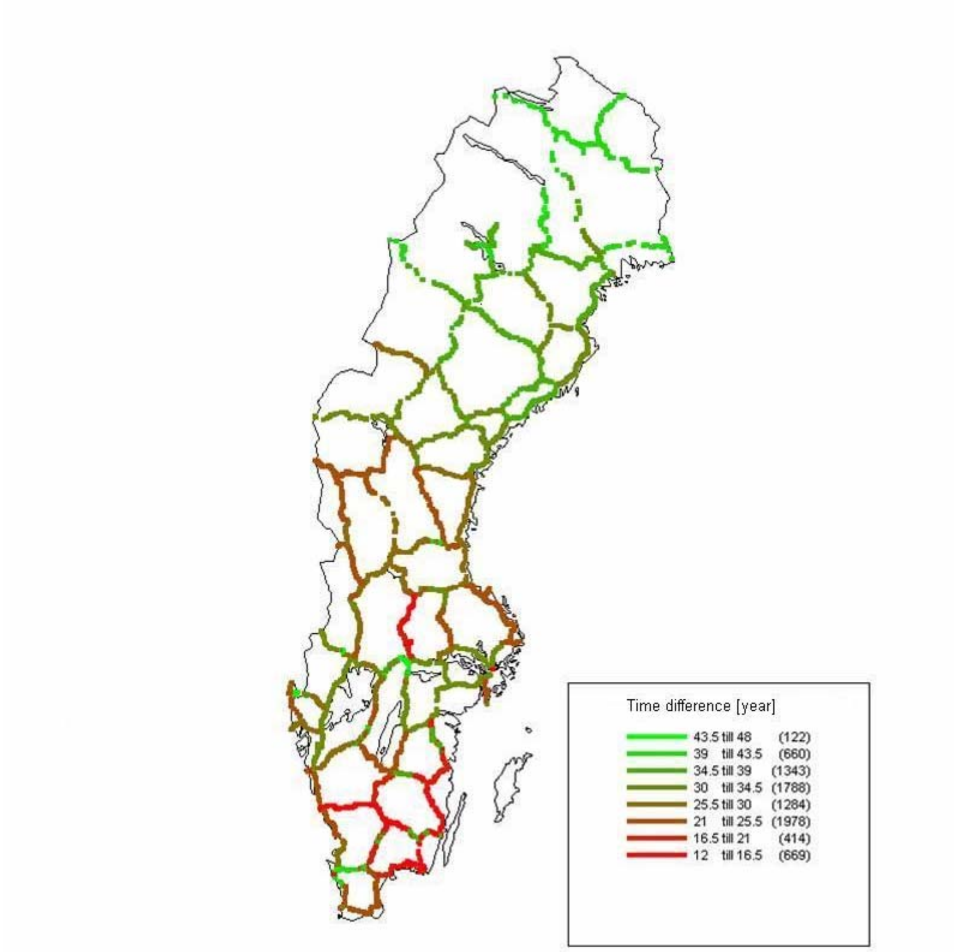

Figure 2.4: The time differences between the second and third precise levellings in Sweden. Unit: mm/year.

All observations used by Vestøl (2005), which have been introduced and discussed above, are finally summarised in Fig. 2.5. The most accurate source of information is still believed to be provided by the tide gauges, but the accuracy of GPS is not far behind. One clear advantage with the latter is that the permanent GPS stations are not limited to the seas. The SWEPOS stations in the central parts of Sweden are a very important complement to the tide gauge and repeated levelling observations.

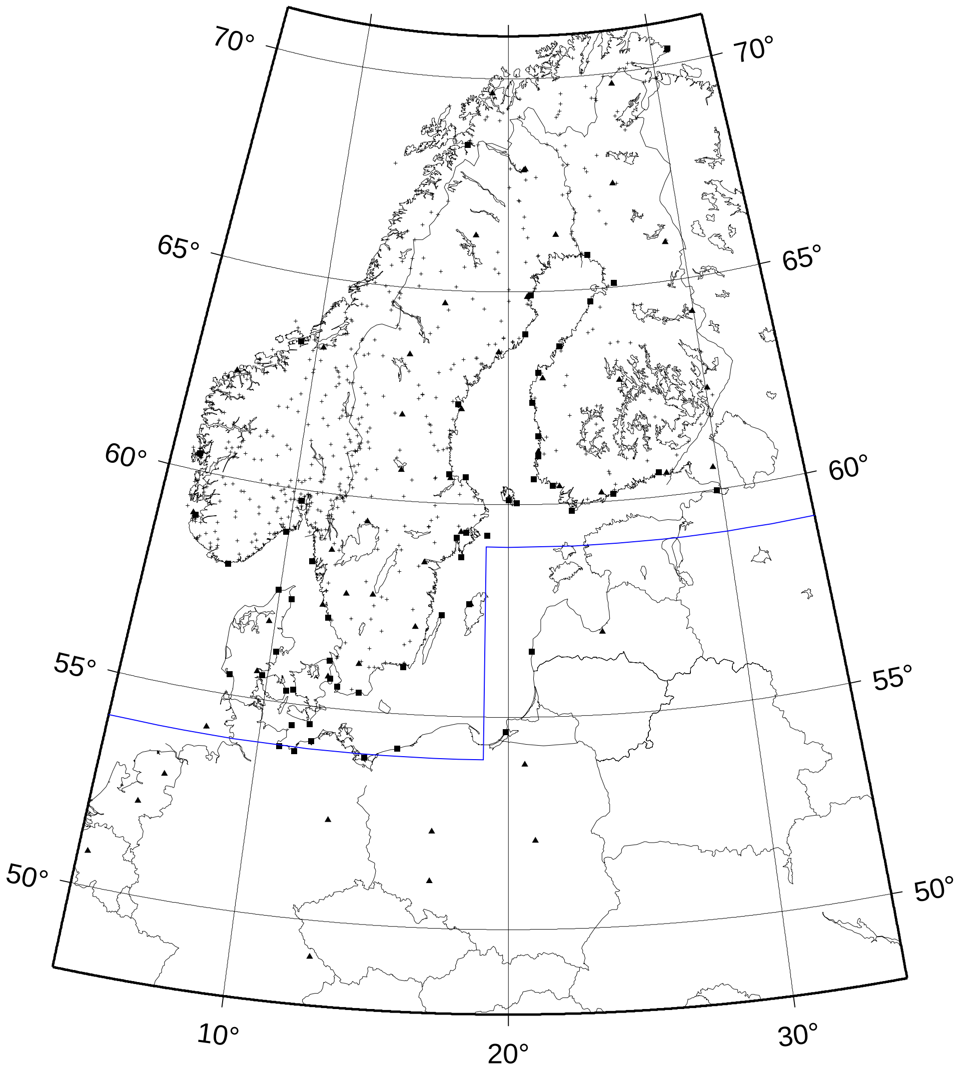

Figure 2.5: Summary of all observations used by Vestøl (2005). The squares symbolise tide gauges, the triangles denote GPS stations and the crosses mean nodal points in levelling lines. Unit: mm/year.

2.2 Vestøl's mathematical model

As mentioned in the introduction, Vestøl (2005) used all the above information to derive a mathematical model for Fennoscandia. The method, which is least squares collocation with unknown parameters (e.g. Moritz 1980), was investigated in this context by Danielsen (1989). The technique was then applied by Vestøl (2002) to estimate the postglacial uplift limited to Norway. Vestøl (2005) finally extended the model to the other Nordic countries and also included GPS observations from the BIFROST project. Below, the method is first summarised and discussed. After that, Vestøl's model is presented and analysed.

2.2.1 Short description of Vestøl's method

A very good treatment of least squares collocation is provided by Moritz (1980), to which the reader is referred for details. Below a short summary is given, mainly to reach a position to be able to discuss the method of Vestøl (2005). The basic observation equation that applies in the present case, sometimes called the mixed model (Koch 1999), reads

$$\boldsymbol{l} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{s} + \boldsymbol{\varepsilon} \tag{2.1}$$

where $\boldsymbol{l}$ is the observation vector, $\boldsymbol{A}$ is the design matrix, $\boldsymbol{x}$ is a vector with unknown parameters, $\boldsymbol{B}$ is a matrix that relates the spatially correlated signals in the vector $\boldsymbol{s}$ to the observations, and $\boldsymbol{\varepsilon}$ is the observation noise vector. It is assumed that the signal $\boldsymbol{s}$ has zero mean and covariance function $\boldsymbol{C}_{ss}\left(\psi_{PQ}\right)$, where the latter depends only on the distance between the two points P and Q (i.e. it is homogeneous and isotropic). The random noise $\boldsymbol{\varepsilon}$ is centred and has the covariance matrix $\boldsymbol{D}$. In addition $\boldsymbol{s}$ and $\boldsymbol{\varepsilon}$ are assumed independent. Now, the least squares collocation solution minimises

$$\boldsymbol{s}^T \boldsymbol{C}_{ss}^{-1} \boldsymbol{s} + \boldsymbol{\varepsilon}^T \boldsymbol{D}^{-1} \boldsymbol{\varepsilon} \tag{2.2}$$

and is provided by

$$\hat{\boldsymbol{x}} = \left(\boldsymbol{A}^T \left(\boldsymbol{B}\boldsymbol{C}_{ss}\boldsymbol{B}^T + \boldsymbol{D}\right)^{-1} \boldsymbol{A}\right)^{-1} \boldsymbol{A}^T \left(\boldsymbol{B}\boldsymbol{C}_{ss}\boldsymbol{B}^T + \boldsymbol{D}\right)^{-1} \boldsymbol{l} \tag{2.3}$$

and

$$\hat{\boldsymbol{s}} = \boldsymbol{C}_{ss}\boldsymbol{B}^T \left(\boldsymbol{B}\boldsymbol{C}_{ss}\boldsymbol{B}^T + \boldsymbol{D}\right)^{-1} \left(\boldsymbol{l} - \boldsymbol{A}\hat{\boldsymbol{x}}\right) \tag{2.4}$$

It should be noticed that the vector $\boldsymbol{s}$ in the above equations only contains the signal in the spatial locations of the observations. This case, which corresponds to pure *filtering* of the observations, can easily be extended to *prediction* in an arbitrary point. The signal s_P in the arbitrary point P is estimated by modifying the cross correlation part of Eq. (2.4) according to

$$\hat{s}_P = \boldsymbol{C}_{s_P s}\boldsymbol{B}^T \left(\boldsymbol{B}\boldsymbol{C}_{ss}\boldsymbol{B}^T + \boldsymbol{D}\right)^{-1} \left(\boldsymbol{l} - \boldsymbol{A}\hat{\boldsymbol{x}}\right) \tag{2.5}$$

where $\boldsymbol{C}_{s_P s}$ is a vector with covariances between the signal in P and in the observation points. If Eq. (2.5) is applied, a continuous surface is interpolated. If the covariance matrix $\boldsymbol{D}$ is non-zero, then the interpolation is smoothing and the observation errors are filtered. For error-free observations an exact interpolator is implied. It should be mentioned that least squares collocation provides the unbiased solution with minimum variance, which is given by

$$\sigma_{s_P}^2 = \boldsymbol{C}_{\boldsymbol{s_P s_P}} - \boldsymbol{C}_{\boldsymbol{s_P s}}\boldsymbol{B}^{\boldsymbol{T}}\left(\boldsymbol{BC}_{\boldsymbol{ss}}\boldsymbol{B}^{\boldsymbol{T}} + \boldsymbol{D}\right)^{\boldsymbol{-1}}\boldsymbol{BC}_{\boldsymbol{ss_P}} + \boldsymbol{HAC}_{\boldsymbol{XX}}\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{H}^{\boldsymbol{T}} \tag{2.6}$$

where

$$\boldsymbol{H} = \boldsymbol{C}_{\boldsymbol{s_P s}}\boldsymbol{B}^{\boldsymbol{T}}\left(\boldsymbol{BC}_{\boldsymbol{ss}}\boldsymbol{B}^{\boldsymbol{T}} + \boldsymbol{D}\right)^{\boldsymbol{-1}} \tag{2.7}$$

The covariance matrix for the unknown parameters is

$$\boldsymbol{C}_{\boldsymbol{XX}} = \left(\boldsymbol{A}^{\boldsymbol{T}}\left(\boldsymbol{BC}_{\boldsymbol{ss}}\boldsymbol{B}^{\boldsymbol{T}} + \boldsymbol{D}\right)^{\boldsymbol{-1}}\boldsymbol{A}\right)^{-1} \tag{2.8}$$

and the cross covariance between the parameters and signals is given by

$$\boldsymbol{C}_{\boldsymbol{Xs_P}} = -\boldsymbol{C}_{\boldsymbol{XX}}\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{H}^{\boldsymbol{T}} \tag{2.9}$$

It is straightforward to derive covariance matrices also for other linear combinations of $\boldsymbol{x}$ and $\boldsymbol{s}_P$ using the law of error propagation. It should be noticed that least squares collocation with unknown parameters implies that the unknown parameters $\boldsymbol{x}$ are first estimated in Eq. (2.3) using standard least squares adjustment with a weight matrix modified to take into account the spatial correlations described by $\boldsymbol{C}_{\boldsymbol{ss}}$. The "residuals" in the observation points $\boldsymbol{l} - \boldsymbol{A\hat{x}}$ are then filtered using Eq. (2.4), which leads to the residuals after the signal part has been removed, i.e. to $\hat{\boldsymbol{\varepsilon}} = \boldsymbol{l} - \boldsymbol{A}\hat{x} - \boldsymbol{B\hat{s}}$. The signal can then be interpolated to arbitrary locations using Eq. (2.5). It should be added that Vestøl does not solve for $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{s}}$ at the observation points using Eqs. (2.3) and (2.4), but prefers the formulation according to Schwarz (1976). This, however, changes nothing in principle: One arrives at exactly the same result in either formulation.

Let us now consider Vestøl's case, in which we have a number of observations that are related to the land uplift. As described in the last section, the *observations* in this case are apparent uplifts at the mareographs, absolute uplifts at the GPS stations and height differences for the levelling lines between nodal benchmarks. Vestøl chooses to model the apparent land uplift by a "systematic" trend part, which is given by a polynomial of degree 5, to which a signal part is added, which is assumed to have the covariance function,

$$
\begin{aligned}
C_{ss}(d) &= 0.1\cdot\left(\frac{100}{400^2}d^2-\frac{8}{400}d+1\right)\ (\text{mm/year})^2 && \text{if } d\le 60 \text{ km}\\
C_{ss}(d) &= 0 && \text{if } d>60 \text{ km}
\end{aligned}
\tag{2.10}
$$

where d is the distance in km. The corresponding correlation length is approximately 25 km and the signal standard deviation is 0.32 mm/year. The reasons for choosing this trend surface together with the covariance function (2.10) will be further discussed below. It might be noted that the choice justifies the use of a homogeneous and isotropic covariance function, which would not be justified in case no trend surface was used. The vector of *unknown parameters* $\boldsymbol{x}$ thus consists of the coefficients of the polynomial and the heights of all involved levelling benchmarks (nodal points). In addition, two more parameters are introduced to relate the absolute uplift $\dot{h}$ provided by GPS to the apparent counterpart $\dot{H}_a$ from the tide gauges; see Ekman and Mäkinen (1996a). The difference is modelled in the following way,

$$
\dot{h}=\dot{H}_a+\dot{H}_e+s\cdot\left(\dot{H}_a+\dot{H}_e\right) \tag{2.11}
$$

where $\dot{H}_e$ is the eustatic sea level rise and s is a scale factor that is used to represent the uplift of the geoid. It is of course somewhat questionable whether the geoid rise can be modelled by a simple linear relationship, but this approximation can be expected to be reasonable; cf. Ekman (1998). More rigorous formulas can be found in Sjöberg (1989). In the present case, the two parameters $\dot{H}_e$ and s are simply estimated together with the other unknowns in Eq. (2.3); see Vestøl (2005). It should further be mentioned that the construction of the design matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ follows from the definition of the observations and the parameters. This is straightforward and need not be discussed here.

An important question is how the observations should be weighted, which amounts to the construction of the matrix $\boldsymbol{D}$ in the above formulas. Vestøl assumes that observations are uncorrelated and then estimates variance components for 10 groups of observations. This means that the dispersion matrix is decomposed as

$$
\boldsymbol{D}=\begin{bmatrix}
\sigma_1^2\boldsymbol{P}_1^{-1} & 0 & \cdots & 0\\
0 & \sigma_2^2\boldsymbol{P}_2^{-1} & & \\
\vdots & & \ddots & \\
0 & & & \sigma_{10}^2\boldsymbol{P}_{10}^{-1}
\end{bmatrix} \tag{2.12}
$$

where $\boldsymbol{P}_i$ is the diagonal weight matrix for observation group *i*. The variance components σ_i^2 are then estimated using the estimator derived by Förstner (1979a,b),

$$\hat{\sigma}_i^2 = \frac{\hat{\boldsymbol{\varepsilon}}_i^{\boldsymbol{T}} \boldsymbol{P}_i \boldsymbol{\varepsilon}_i}{r_i} \tag{2.13}$$

which can be shown to be a Best Quadratic Unbiased Estimator (BQUE). Here $\hat{\varepsilon}_i$ are the estimated residuals for group *i* and r_i are the corresponding local redundancies; see and Koch (1999) for further details. After the variance components have been estimated, they are introduced into Eq. (2.11) and the whole procedure is repeated until convergence (if it converges). It should also be mentioned that the estimation of variance components is combined with a test of outliers; see Vestøl (2005). The test statistic is the estimated outlier divided by its standard error. If this quantity is larger than 3, the observation is considered to be contaminated by a gross error.

Above most aspects of the method used by Vestøl (2005) has been summarised. Let us now consider one important part that has not been mentioned so far, and which was not understood at first by the present authors and which has caused a lot of confusion. As explained above, the land uplift is modelled by a trend surface (represented by a 5^{th} degree polynomial), to which the estimated signal is added. One major problem here is that it is not possible to compute the trend outside the given observations. The polynomial very likely will start to behave violently when moving too far. To avoid this problem, Vestøl (2005) limits the use of least squares collocation with unknown parameters to the estimation of land uplift values at the observation points only. A *completely different gridding algorithm* is then used to produce the final grid. The grid values are computed from the estimated uplift at the observation points as the weighted mean (inverse distance weighting) of maximally four observations using a search algorithm. The closest observation in each of four quadrants is chosen in case it is situated within 120 km from the grid point. This means that in case only one observation is within 120 km, the grid value becomes equal to the "nearest neighbour". A single observation therefore produces a *cylinder with 120 km radius*. If no observation is within 120 km, the grid point is not defined. Thus, we repeat, the grid is *not* produced by adding the trend surface and the predicted signal from Eq. (2.5) in each grid point. It is similarly a misunderstanding that the correlation length of Vestøl's method is 120 km. The latter figure is only used in the search algorithm. As mentioned above, the correlation length of the covariance function (2.10) is equal to approximately 25 km. The above information has been confirmed by Vestøl (personal communication).

As mentioned in Section 2.1, it is possible to estimate the land uplift from non-repeated levelling. It is important to notice that this requires that the levelling lines form loops or are connected in some kind of structure involving lines from different epochs. Otherwise, it is not possible to extract the land uplift. Let us elaborate a little on this point. Imagine four levelling lines forming a star according to Fig. 2.6, where one of the benchmarks is fixed to an arbitrary height. It is then obvious that we have four observations and the same number of parameters. Consequently, it is not possible to obtain any information concerning the uplift. No matter how large the uplift is, it is always compatible with the observations. What happens when the uplift field changes is simply that the heights adjust accordingly. Consider on the other hand the situation in Fig. 2.7, which contains one redundant observation. Here a change in the uplift field affects the observed lines, which means that the observations can be used to determine the uplift. The measurements are related to the uplift difference between benchmark 1 and 4. If all lines in the loop have been observed at different epochs, one equation is provided with three unknown uplift differences (e.g. two in the east-west and one in the north-south directions) and so on.

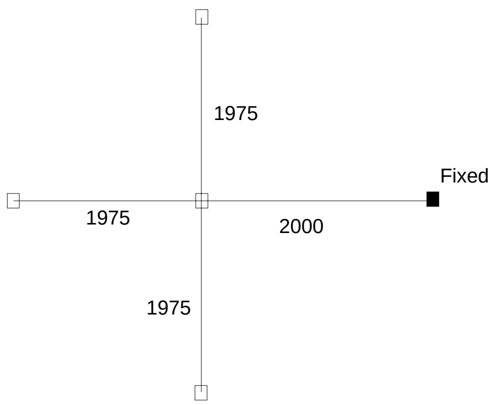

Figure 2.6: A levelling network that does not contain any information on the land uplift.

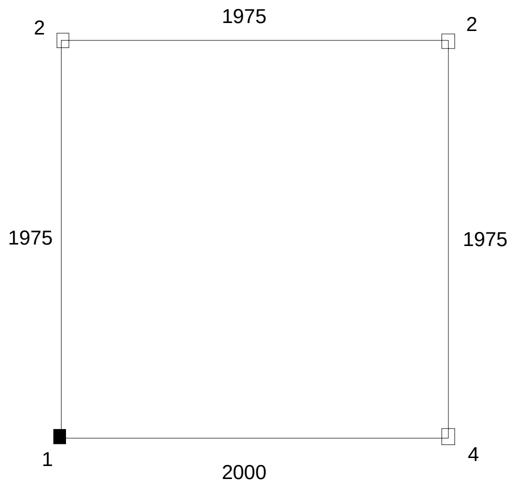

Figure 2.7: Illustration of a loop that contains information on the land uplift.

Consider now the network in Fig. 2.8. Here the two loops provide two equations involving the uplift differences in the north-south and east-west directions. Since the uplift itself is also provided by a tide gauge in the fixed benchmark 1, the above structure may be used to estimate the uplift (not only differences) in case it is modelled by an inclined plane. It should be noticed that the network does not provide any redundancy. Notice further that the loose end to benchmark 8 does not add any information concerning the uplift, but since the uplift plane has been determined by the two loops and the tide gauge, it can be utilised to obtain the uplift for correction of the observations to the reference epoch. The situation is exactly parallel when the Norwegian levelling lines are used to determine the uplift. In this case, however, the land uplift is modelled by a fifth degree polynomial, to which a signal estimated by least squared collocation is added. The levelling lines forming loops in the inland parts of Norway helps to determine the uplift, but the many loose ends in the coastal regions do not contain any uplift information at all. In this case, the uplift field stems from tide gauges, GPS stations and levelling loops nearby. Naturally, this creates a rather unsatisfying situation at the "loose" or open lines, since the uplift is modelled by a fifth degree plus a signal. As was mentioned above, it is a well known behaviour of a higher degree polynomial that it starts to deviate violently outside the area with observations. As the open levelling lines do not constrain the uplift in any way (the uplift is only used there), it is questionable how good the resulting polynomial extrapolation is. It is true that the uplift field is also modelled by a signal part, but this does not help much. The covariance function must be chosen to be representative for the difference with respect to the *constrained* polynomial inside the observation area. In addition, the covariance is low when the loose ends (open lines) are long, which means that the estimated signal can be expected to be small. When the outermost loose end point gets further than 60 km from its nearest neighbour, the covariance is identically zero for Vestøl's function in Eq. (2.10), which means that the signal vanishes. This happens for several stations along the Norwegian coast.

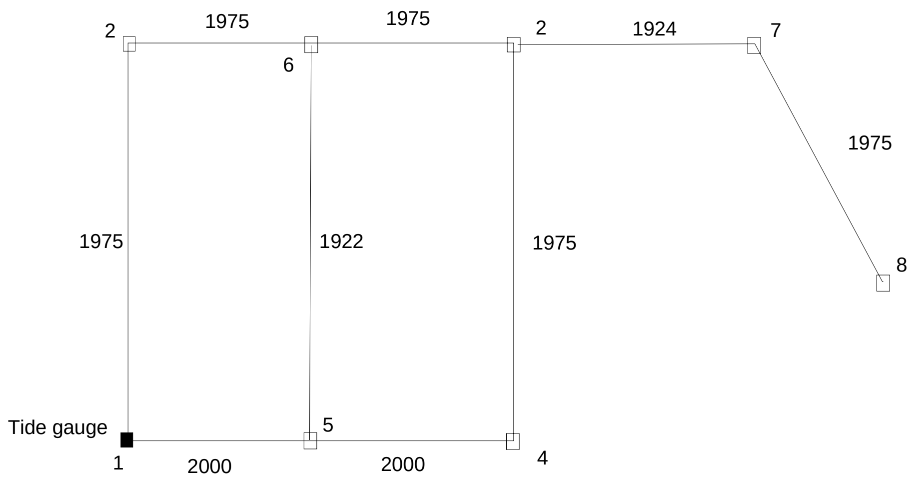

Figure 2.8: Illustration of a network for which an inclined plane might be used to represent the uplift.

2.2.2 The model in gridded form

In this subsection some interesting numerical results from the computation of Vestøl's model are first presented and discussed. Here only a few key issues are considered which are important for the choice of uplift model for RH 2000. The reader is referred to Vestøl (2005) for more details. After that, the final gridded model is presented and analysed. The subsection ends with a discussion of some of the shortcomings of the model.

As mentioned in the last subsection, Vestøl estimates variance components using the technique presented by Förstner (1979a, b). Some information for the 10 observation groups are summarised in Table 2.2. As can be seen, the process has not been iterated until convergence. It is further somewhat uncertain how accurate the estimates of the variance components are. It would indeed be helpful with confidence intervals, but one drawback with the Förstner method is that no standard errors are obtained for the estimated components. Another question is how the estimation of variance components is related to the choice of trend surface and signal covariance function. How are the variance components affected by changes in the covariance function? Thus, it seems uncertain how the variance components in Table 2.2 should be interpreted. For instance, should we say that the tide gauge uplifts are really as accurate as 0.1 mm/year? Is the first Swedish levelling as good as indicated in Table 2.2? The estimated standard error of unit weight in Table 2.1 is more than twice as large. On the other hand, the results in Table 2.2 indicate that the weighting seems reasonable. The apriori standard errors looks approximately realistic, and the iteration in question indicates that nothing revolutionary happens in the estimation. However, too far-reaching conclusions regarding the accuracy of the different observation groups should be avoided. This means that it is wise to be a little sceptical concerning the accuracy of the resulting uplift model. Close to the tide gauges, the standard error of the estimated uplift will be close to 0.1 mm/year. These figures entirely depend on the apriori standard error assumed for the tide gauges. According to Ekman (1996), a value of 0.2 mm/year would be more justified. With what certainty can we say that 0.1 mm/year is true and 0.2 mm/year false?

Table 2.2: Observation groups, apriori standard errors and variance components for the last iteration. From Vestøl (2005).

#	Description	Apriori standard errors in $\boldsymbol{P}_i$	σ_i	$\hat{\sigma}_i$
1	Norwegian levelling 1916-1972	1.34 $\text{mm}/\sqrt{\text{km}}$	1	0.993
2	Norwegian levelling 1972-2003	1.12 $\text{mm}/\sqrt{\text{km}}$	1	0.994
3	Finnish 1st levelling	1.07 $\text{mm}/\sqrt{\text{km}}$	1	1.006
4	Finnish 2nd levelling	0.85 $\text{mm}/\sqrt{\text{km}}$	1	1.016
5	Finnish 3rd levelling	0.80 $\text{mm}/\sqrt{\text{km}}$	1	0.983
6	Swedish 1st levelling	2.04 $\text{mm}/\sqrt{\text{km}}$	1	1.000
7	Swedish 2nd levelling	1.41 $\text{mm}/\sqrt{\text{km}}$	1	0.998
8	Swedish 3rd levelling	1.10 $\text{mm}/\sqrt{\text{km}}$	1	1.005
9	Permanent GPS stations	1.51 times the standard errors estimated by Lidberg (2004); cf. the discussion in Sect. 2.1.	1	0.992
10	Tide gauges	0.10 mm/year	1	1.010

Let us turn now to the parameters for the difference between absolute and apparent uplift. The estimated parameters and standard errors obtained by Vestøl (2005) are the following:

$$\begin{aligned} \hat{\dot{H}}_e &= 1.32 \pm 0.14 \ \text{mm/year} \\ \hat{s} &= 6 \pm 2\ \% \end{aligned} \tag{2.14}$$

The eustatic sea level rise agrees well with what have been obtained by others, for instance the estimate $\dot{H}_e = 1.05$ mm/year of Lambeck et al. (1998); see further Ekman (2000). The scale factor is exactly the same as in Ekman and Mäkinen (1996a) for the centre of the uplift area, but Ekman (1998) uses the same scale factor for the whole of Fennoscandia. The corresponding apparent uplift values in the permanent GPS stations are presented in Fig. 2.9.

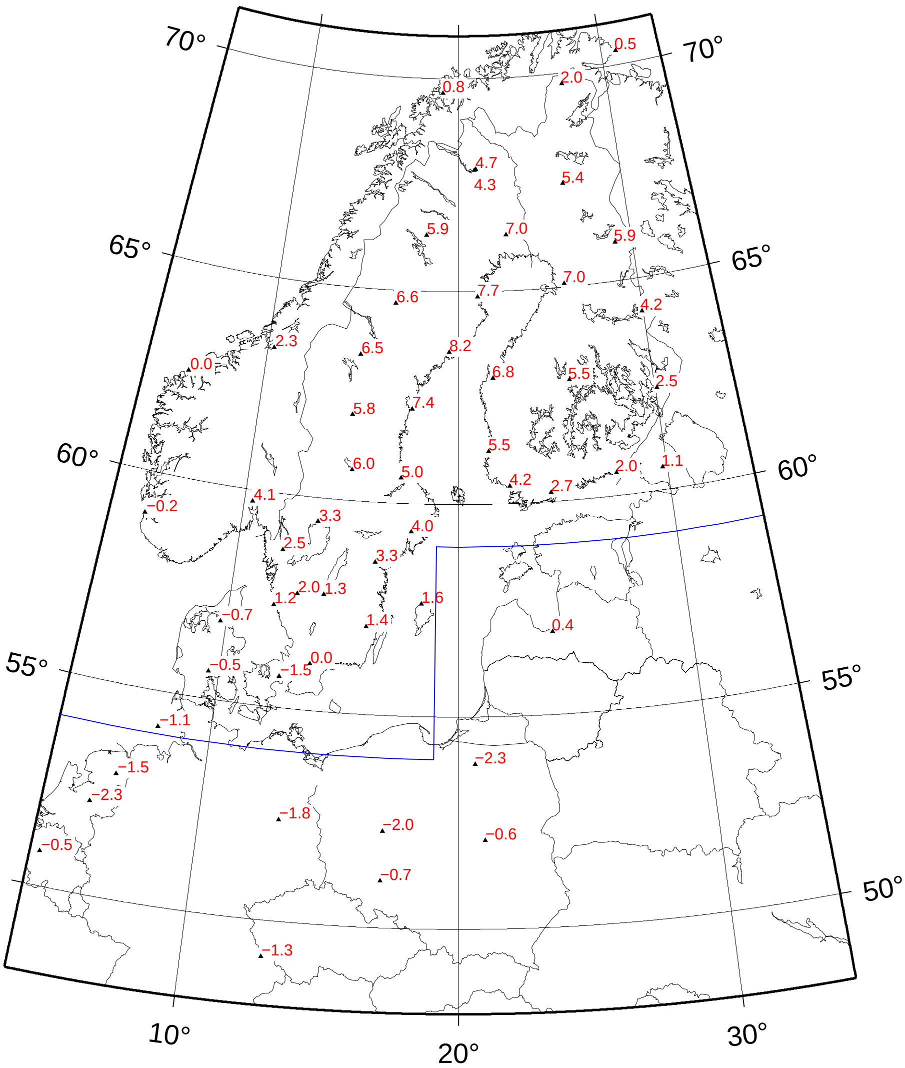

Figure 2.9: Apparent land uplift values at the *GPS-stations* calculated using the linear model parameters estimated by Vestøl (2005). Unit: mm/year.

It is not the purpose of this report to present all practical details from Vestøl (2005). Let us comment, however, on two of the most notable gross errors that were detected and removed. The most important one for our concern can be seen by comparing Figs. 2.1 and 2.9. It is clear that the land uplift maximum is situated further to the north in the tide gauge case as compared to the GPS case. The apparent uplift from the mareograph in Furuögrund (8.75 mm/year) is approximately 1 mm/year larger than the same quantity in the permanent GPS station in Skellefteå (7.7 mm/year). As the latter is consistent with the three Swedish precise levellings, the tide gauge observation shows up as a clear outlier in the gross error detection. This means that Vestøl's model has its centre to the south compared to the models that include Furuögrund, for instance Ekman (1996). This feature is obviously important for the computation of RH 2000 and needs to be considered when the final uplift model is chosen. Another notable gross error is the mareograph in Oslo (4.1 mm/year). We believe that this case is not as clear as the first, since the tide gauge and GPS observations now agree perfectly. According to Vestøl, however, these observations are contradicted by numerous levelling lines, which imply that the tide gauge observation is marked as an outlier. As the largest uplift differences in this case occur in Norway, the exclusion is perhaps not too crucial in Sweden. In any case, the two outliers should be kept in mind. It should finally be mentioned that no GPS observations are excluded as outliers, but 43 levelling lines are rejected; see Vestøl (2005) for details.

It is now time to take a look at the Vestøl (2005) model in its gridded form. It is here important to remember that Vestøl used least squares collocation with unknown parameters only to estimate the land uplift in the observation points. After that, an independent gridding algorithm is taken advantage of to produce the grid; see the discussion in the last subsection. The model is presented by a wireframe plot in Fig. 2.10 and by contour lines in Fig. 2.11. Notice that the model is undefined for all grid points further than 120 km from the closest observation point. It is clear from Fig. 2.10 that the model is rather rough, also in some of the more central parts. This feature can also be discerned by studying the contour lines in Fig. 2.11, which are very curvy. It should be noted that the model is undefined for large areas, particularly at the south-east side of the

Baltic Sea. Furthermore, as was explained in Subsection 2.2.1, cylinders are formed around the isolated GPS observations to the south. This entirely depends on the interpolation method that was used in the gridding. The same effect can also be spotted at the borders of the model (along the coast of Norway and outside the Finnish-Russian border).

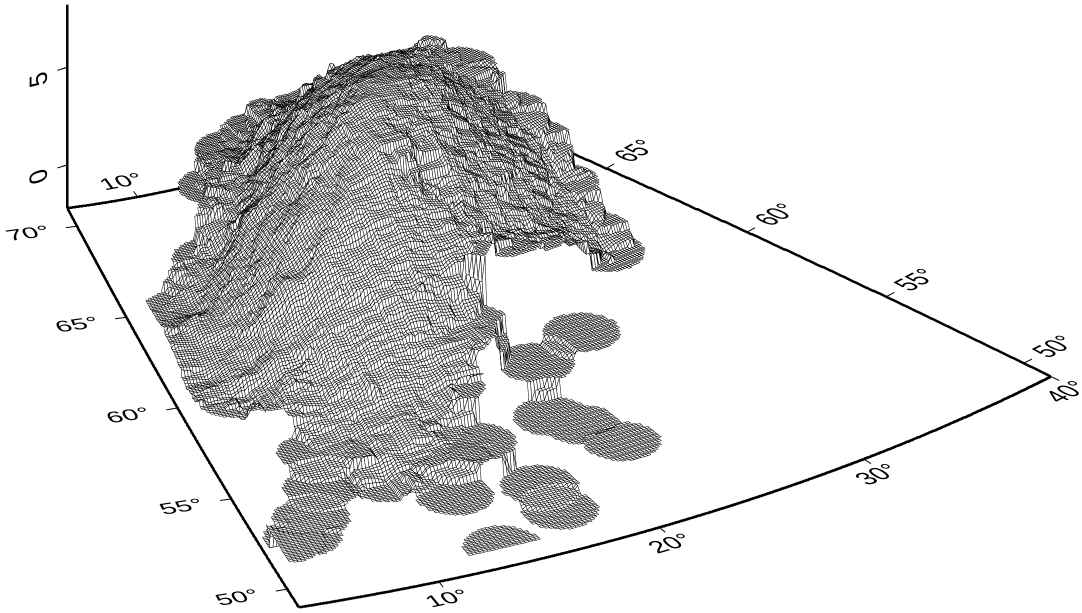

Figure 2.10: Apparent land uplift from Vestøl's grid model. Unit: mm/year.

Let us now study how the model fits the given observations. For all uplift models that will be studied in this report, comparisons are made with the given tide gauge and GPS observations. It is much more difficult to summarise and visualise the residuals for the levelling lines/sections. As it is believed that GPS and tide gauges are most important, at least in Sweden, we feel content with presenting statistics and residuals only for the latter observation types. Now, the statistics for Vestøl's grid model can be found in Table 2.3. The tide gauges are presented both with and without the two outliers discussed above, and the GPS statistics are considered for all 55 GPS stations provided by Lidberg (2004) as well as for only the SWEPOS stations. The reason for including the last case is that the SWEPOS stations have low standard errors and are important for the present purpose. They provide the most reliable information in the central parts of Sweden.

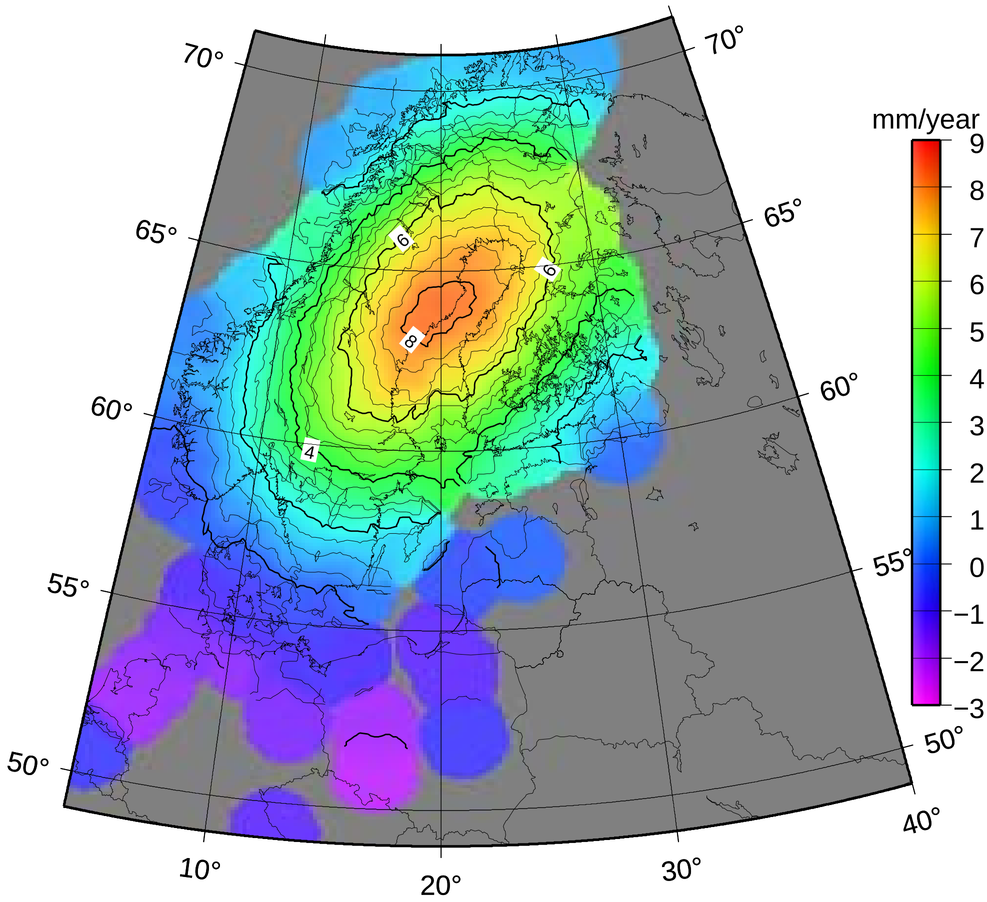

Figure 2.11: Contour lines for the apparent land uplift of Vestøl's grid model. Zero uplift is plotted where the model is undefined. Unit: mm/year.

Table 2.3: Statistics for the apparent uplift residuals for Vestøl‘s grid model. The maximum for “All tide gauges” is given for both the outlier stations discussed in the text (Furuögrund/Oslo). Unit: mm/year.

Observations	#	Min	Max	Mean	StdDev	RMS
All tide gauges	58	-0.19	0.88/1.20	0.04	0.20	0.20
Cleaned tide gauges	56	-0.19	0.18	0.00	0.08	0.08
All GPS	55	-1.27	1.53	-0.02	0.45	0.45
SWEPOS GPS	21	-0.56	0.31	-0.03	0.23	0.23

It is clear that Vestøl model behaves as could be expected. It fits extraordinarily well to the tide gauges, which of course depends on the very high weight given to these observations; cf. Table 2.2. The fit to the SWEPOS stations is further good and the accuracy degrades when all GPS stations are considered, exactly as indicated by the standard errors of Lidberg (2004). As discussed above, the two outliers differ considerably from the model, approximately 1 mm/year in both cases.

We now take a look at the estimated standard deviations for the estimated apparent land uplifts. As it is not easy in practice to propagate the standard errors through the independent gridding process, a nearest neighbour plot of the observation point standard errors is presented in Fig. 2.12. Very low values occur close to the tide gauges (cf. Fig. 2.1). This depends on the assumed apriori standard errors; see the discussion at the beginning of this subsection. It is difficult to judge whether the latter are realistic or not. It is more surprising, at least to the present authors, that comparatively low standard errors are also obtained for areas with only repeated levelling. The standard errors are between 0.12–0.2 mm/year more or less in the whole of Finland, which obviously depends on the contribution from repeated precise levelling. In Sweden, the standard errors are typically between 0.2-0.3 mm/year. This shows that all the available information improves the standard errors significantly, compared to what could be expected in case only one single observation type is available; cf. the error propagation made at the end of Section 2.1. However, as was mentioned in this discussion, in case systematic or gross errors are present, the estimated standard errors are very likely to be too pessimistic. Another observation that can be made in Fig. 2.12 is that the quality of the permanent GPS stations is low at the southern parts of the area (continental Europe). It can finally be seen that the standard errors are high along the Norwegian coast, which mainly depends on the "loose ends" that were discussed at the end of Subsection 2.2.1. In these points, the uplift is extrapolated using a fifth degree polynomial, which is also reflected in large standard errors close to 0.5 mm/year.

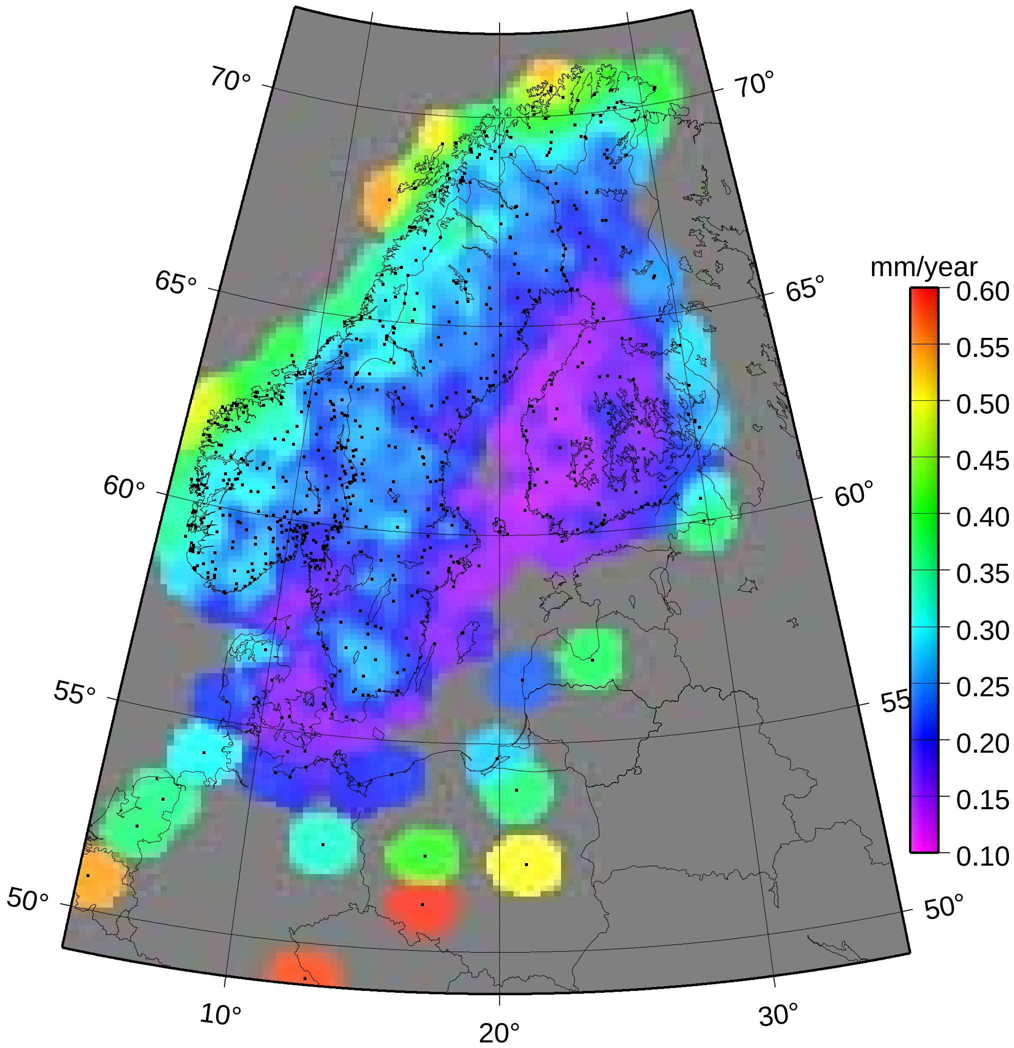

Figure 2.12: Nearest neighbour plot of the estimated standard errors for Vestøl's model in the observation points. Unit: mm/year.

To sum up, Vestøl's model agrees well with the given observations, which could be expected from the assumed standard errors. On the negative side, the basic problems are the following:

- The model is not defined for the whole Baltic Levelling Ring network and needs to be extended.
- The "cylinders" in the outskirts of the model are disturbing.
- It is rough, particularly in Norway and Finland (see Fig. 2.10). Considering the high flexural rigidity of the crust, it is not likely that real postglacial land uplift behaves in this way. It is clear that it is difficult to choose the trend surface, the covariance function and the apriori standard errors in such a

way that the mathematical model becomes a realistic representation of the uplift field. There is no doubt that Vestøl (2005) has done a great work in synthesising all the given information, but it might be discussed if it would not be physically more realistic to represent the model in a smoother way. Of course, smoothing will not automatically give us the true uplift, but to the authors' opinion, it will result in a more accurate land uplift model, due to the reduction of high-frequency observation errors.

The first problem is most crucial for the construction of a land uplift model for the computation of RH 2000. To be able to adjust the whole Baltic Levelling Ring network, Vestøl's model obviously needs to be extended to a very large area. As Vestøl (2005) utilises a polynomial as trend surface, it is not possible for him to apply least squares collocation with unknown parameters for extrapolation. The polynomial would go crazy (so to speak). One suitable thing to do here is obviously to replace the use of a polynomial trend surface by a remove-compute-restore procedure with respect to a given model, for instance the geophysical model of Lambeck et al. (1998). This means that we would utilise least squares collocation to model the difference from Lambeck's model. After the grid surface has been estimated by least squares collocation, Lambeck's model is restored. This procedure has many advantages and certainly makes it possible to extrapolate the residual field without problems. However, as no time was available to realise this modification, we had no choice other than doing as good as possible with the models and observations at hand (at the end of 2004). Thus, several methods were tested to extend Vestøl's gridded model to the whole Baltic Sea area. As the best geophysical model at the time was considered to be Lambeck's model, it was chosen to supply the missing information. These investigations are presented in Section 3.1.

The second problem in the above list shows that Vestøl's independent gridding algorithm is not optimal for the task. It therefore seems suitable to replace it. As the model of Vestøl (2005) is available also in the observations points, it is possible for us to consider Vestøl's model as defined in these points only, and neglect the Vestøl grid. Different methods for gridding and extension of Vestøl's *point* model are consequently investigated in Section 3.2. It should be mentioned that, since Vestøl (2005) does not mention anything about a separate gridding algorithm, the authors did not understand at first that such a method had been used. During this phase, the investigations presented in Section 3.1 were made. As soon as it was realised that the "cylinders" (or "bubbles" as Engsager calls them) were caused by the gridding, it seemed evident to start from the observation points. However, it is nevertheless believed that it is instructive to present the work made on Vestøl's gridded model, which is the reason for including these investigations in Section 3.1.

To sum up, a more extensive model is needed to extrapolate Vestøl's model outside the original observations. As no other observations are available, it seems like the best option is to utilise the geophysical model of Lambeck et al. (1998). This model is investigated in the next section.

2.3 Evaluation of Lambeck's geophysical model

As is revealed by the discussion of the third item above, a basic problem with a mathematical model is that it is uncertain how the data should be interpolated. In the case of a geophysical model, a physically meaningful interpolation method is supplied; at least as far as the model is realistic. It might therefore be better for us to use the geophysical model of Lambeck et al. (1998) by itself. As was discussed in the last section, this model is also considered as the best alternative for extrapolation of Vestøl's model. In both ways, it is important to know the accuracy of the model. As explained in the introduction, Lambeck's model has been tuned to the same tide gauge data as used by Vestøl (2005), which was taken from Ekman (1996); see Section 2.1 above. In addition, ancient shore line observations and other geophysical knowledge have been utilised in the construction of the model (Lambeck et al. 1998). It is the purpose of this section to evaluate Lambeck's model using the tide gauge and GPS observations.

In the introduction it was mentioned that that the geophysical model of Lambeck et al. (1998) was chosen for future work by the working group for height determination within the Nordic Geodetic Commission (NKG). Since Kurt Lambeck did not agree to make his 1998 model available in digital form, the working group had no choice but to digitise the model from the paper publication of

Lambeck et al. (1998). This task was performed in parallel by the National Land Survey of Sweden (Lantmäteriet) and the Finnish Geodetic Institute (FGI). The two groups arrived at more or less the same result. The original figure from Lambeck et al. (1998) is shown in Fig. 2.13, while the digitised version is presented in Figs. 2.14 and 2.15. Unfortunately, it was found that the digitised contour lines were not compatible with the given model values in the tide gauges, which were explicitly presented in Table 1 of the same publication. It thus seems that different versions of the model were presented in different parts of Lambeck et al. (1998). In what follows, Lambeck's digitised model will be assumed as Lambeck's model and no reference will be made to Table 1 in Lambeck et al. (1998). It is thus simply assumed that the digitised version constitutes the "real" Lambeck model.

Another assumption concerns the uplift values outside the -1.0 mm /year curve in Fig. 2.13. It is here rather arbitrarily assumed that the minimum apparent uplift is -2.0 mm/year, which means that the splines used to convert the digitised contour lines to grid values are only extended until this value. If the eustatic sea level rise of 1.05 mm/year is considered (Lambeck et al. 1998), this corresponds to a land sinking of 0.95 mm/year with respect to the geoid in the outer parts of the model. Of course, this situation is not realistic as far as the "real" land sinking dies out pretty soon further away from the uplift area. However, if it is considered that the model is only to be applied outside the -1.0 mm/year curve (apparent uplift) in the Netherlands, Germany and Poland, then it appears that the approximation is not so bad after all. In any case, it is not clear how to extend Lambeck's model beyond the -1 mm/year curve in Fig. 2.13. There is simply not enough information available.

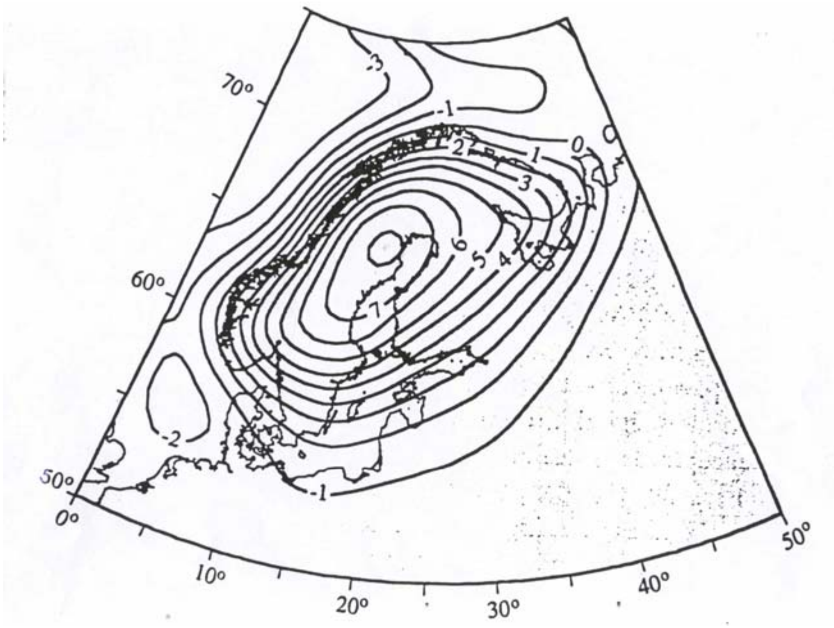

Figure 2.13: Contour lines of the land uplift model published by Lambeck et al. (1998). Apparent uplift. Unit: mm/year.

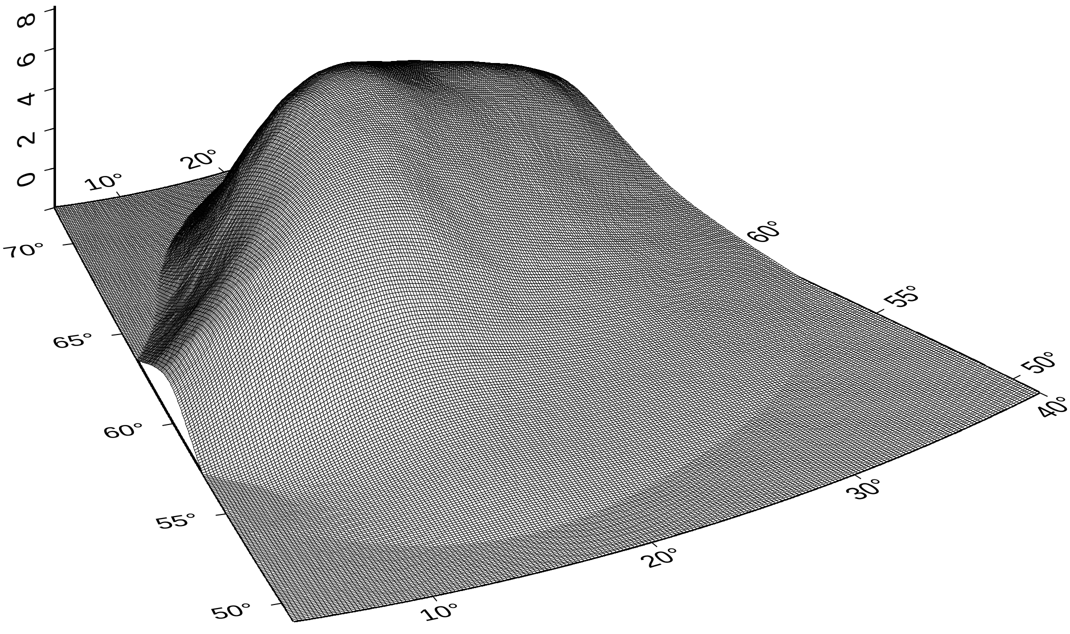

Figure 2.14: Apparent land uplift from Lambeck's model. Digitized version of Lambeck et al. (1998) with -2.0 mm/year as minimum value. Unit: mm/year.

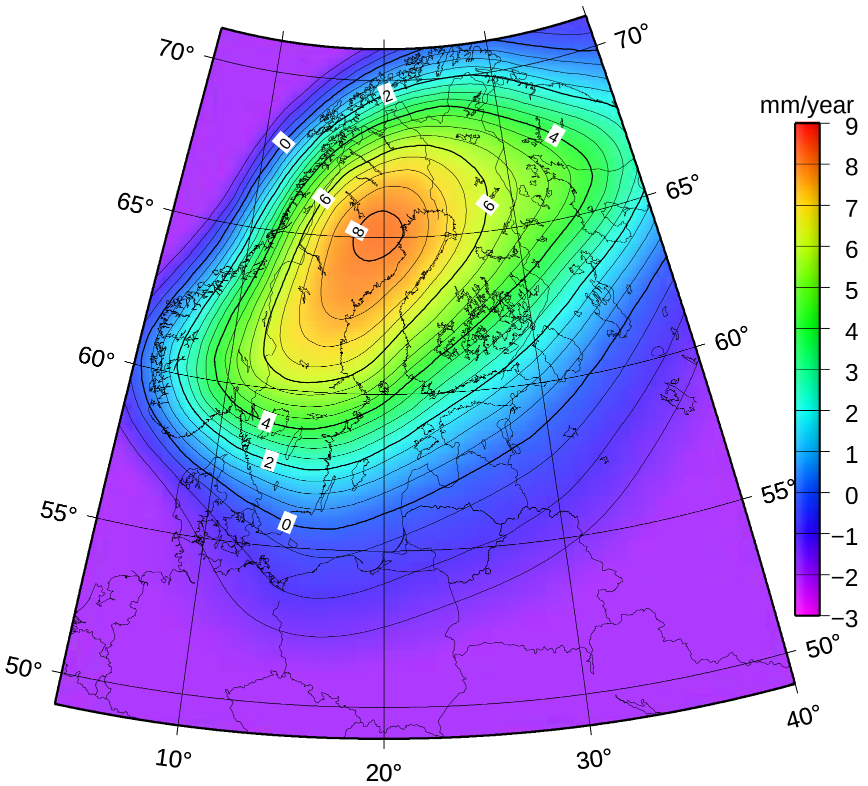

Figure 2.15: Contour lines for the apparent uplift of Lambeck's digitised model. Unit: mm/year.

Let us now evaluate Lambeck's model in the mareograph and GPS stations in exactly the same way as for Vestøl's model in Subsection 2.2.2. The corresponding statistics are presented in Table 2.4 and the residuals in question are plotted in Fig. 2.16. It can be seen that the model fits comparatively well to the tide gauges, even though some rather large deviations can be found; cf. also Lambeck et al. (1998). Again, the two most problematic areas in the tide gauge case are close to the land uplift maximum and in the Oslo region. The model is 0.7 mm/year too low in Furuögrund, but now significant outliers occur also on the Finnish side of the Gulf. Lambeck's model fits almost perfectly in Oslo, but is on the other hand 1.5 mm/year too high in Nevlunghavn immediately south of Oslo. It also fits rather poorly with the mareograph in Smögen (on the Swedish coast south of Oslo). It is true that the original time series in Nevlunghavn only was 40 years (Ekman 1996), which might have caused a large residual, but the uplift should nevertheless be considerably more accurate than the large residual obtained. Thus, according to the mareograph observations, the uplift gradient is very strong in the Oslo Fiord. This feature seems difficult to model using both geophysical and mathematical models. The hand plotted model in

Ekman (1996) reproduces the Oslo observations much better. Except for the problems discussed above, the fit to the tide gauges is promising, which can be seen in case Fig 2.16 is carefully studied. However, the most important thing to notice in Fig. 2.16 is that Lambeck's model disagrees with the inland GPS observations in Sweden. From Kiruna in the north to Jönköping in the south, Lambeck's model is systematically approximately 1.0–1.5 mm/year too high. This might be explained by that no GPS observations were utilised to tune Lambeck's model, but it is nevertheless a little surprising that the model is so bad for the middle parts of the country. After all, ancient shore line observations were also taken advantage of by Lambeck et al. (1998). This deficiency makes Lambeck's model ill-suited to be used as land uplift model in the final computation of the third precise levelling in Sweden, which covers almost the whole country; see Fig. 2.3.

Table 2.4: Statistics for the apparent uplift residuals for Lambeck's model. Unit: mm/year.

Observations	#	Min	Max	Mean	StdDev	RMS
All tide gauges	58	-1.50	1.03	-0.01	0.46	0.46
Edited tide gauges	56	-1.50	1.03	-0.02	0.46	0.46
All GPS	55	-1.64	1.46	-0.25	0.71	0.75
SWEPOS GPS	21	-1.57	0.37	-0.49	0.54	0.73

To sum up, Lambeck's geophysical model is very smooth. The main problem is that it fits poorly with the GPS observations in the central parts of Sweden, which disqualifies it from being used as land uplift model for RH 2000. This shows that it is important that a geophysical model is tuned to *all available geodetic observations*, which is not yet the case with any of the available geophysical models. Thus, it is concluded that the best option in the present case is to start from the mathematical model of Vestøl (2005) and use Lambeck's model *only* as a reference in a remove-compute-restore approach to interpolate and extrapolate Vestøl's grid or point values. This is the main topic of the next chapter.

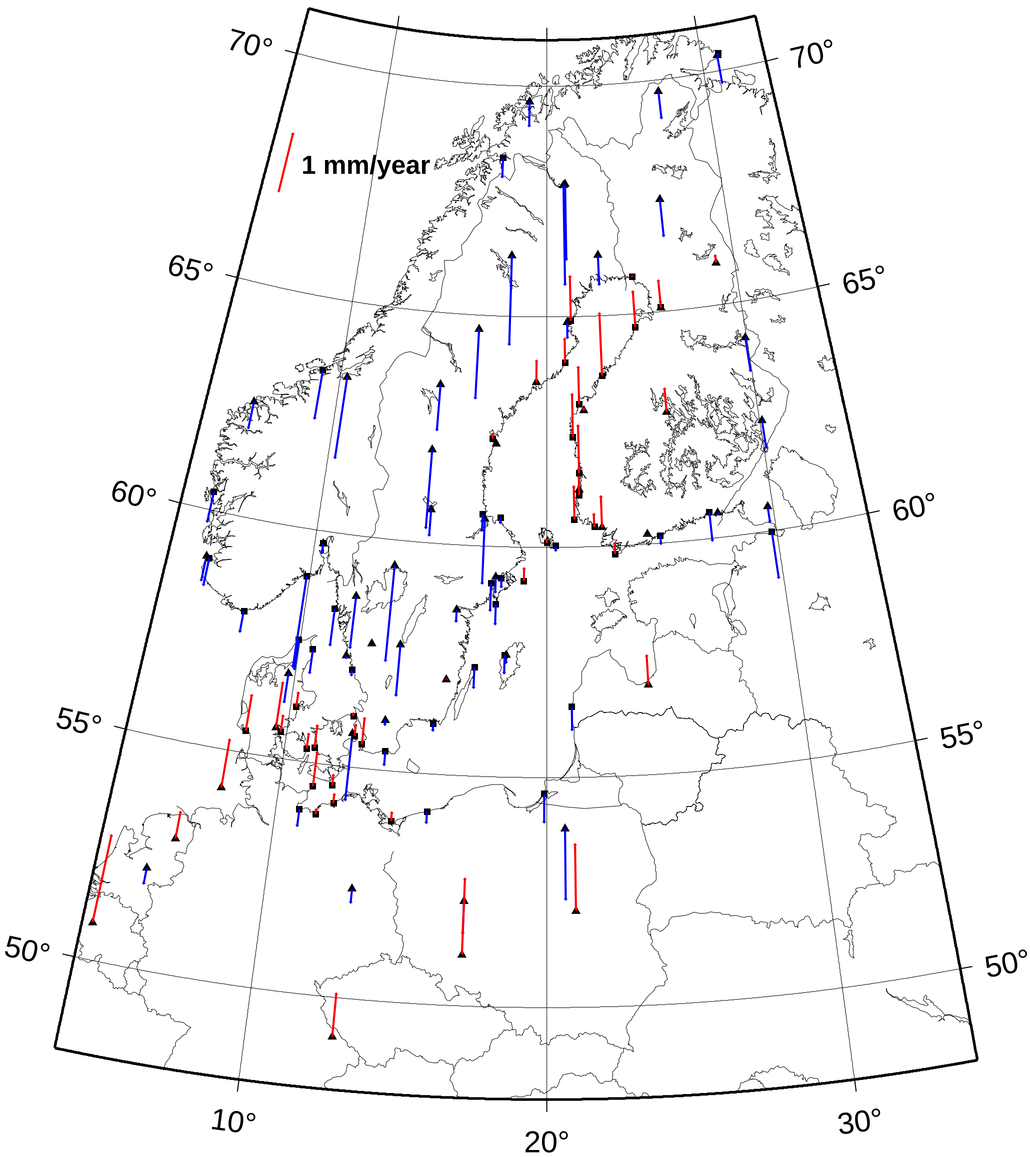

Figure 2.16: Difference between the GPS and tide gauge observations and Lambeck's model.

3. Combination of the Lambeck and Vestøl models

In the last chapter, it was concluded that the best option at the present time to construct a land uplift model for RH 2000 is to start from Vestøl's mathematical model in the central parts of the area and take advantage of Lambeck's geophysical model outside that. The latter might also favourably be used as a reference model for interpolation and extrapolation. However, it is far from certain how the two models should best be treated and combined. This is the main topic of the present chapter. It should be stressed that the severe time limitations for the finalisation of the RH 2000 project implied that it was not possible to construct new geophysical models or improved versions of Vestøl's model. We simply had to do the best under the circumstances using the available information.

In the last chapter it was explained how Vestøl (2005) uses least squares collocation with unknown parameters to estimate the land uplift in the observation points. A completely different method is then applied for gridding. Since some of the problems with Vestøl's model are caused by the latter technique, for instance the staircase cylinders showing up at the borders, it would be logical to start from the uplift in the observation points. However, as it was not known at first that Vestøl (2005) had applied an independent gridding method a lot of work was made starting from the gridded model. This work is summarised in Section 3.1. To include these investigations has the advantage that the road followed by the authors is illustrated, which explains some of the features of the final result that otherwise would seem arbitrary. In the next section, it is investigated how Vestøl's point model should be interpolated and extrapolated, using Lambeck's model as reference in a remove-compute-restore approach. After a suitable interpolation method has been chosen, the chapter finishes with a section that investigates how the different interpolation techniques affect the closing errors around the Gulf of Bothnia and the Baltic Sea.

3.1 Some extensions of Vestøl's grid model

It is the main purpose of this section to present some extensions of Vestøl's grid using Lambeck's model. Since the final model will be derived starting from Vestøl's point values, no comparisons are here made with the available observations. The different attempts are only presented using so called wireframe plots. It is believed that these plots highlight some important points concerning how Vestøl's model should best be treated.

The most straightforward and simple way to extend Vestøl's grid is to take the uplift from Lambeck whenever Vestøl is undefined. This strategy results in the model illustrated in Fig. 3.1. Inside the definition area, it is exactly the same as in Figs. 2.10 and 2.11. It can be seen that Lambeck's model agrees reasonably well on an average, but the cylinders are now more disturbing. The cylinders at the Norwegian and Finnish borders and in the central parts of Europe imply jumps on the 1-2 mm/year level, which looks bad (to say the least). It might be thought that it is possible to diminish the magnitude of the jumps by adjusting Lambeck's model up or down, but this is only marginally so. The GPS observations in the central parts of Europe (south of latitude 54°) disagree too much internally. In addition, the jumps in Norway and in the southern parts of Finland are impossible to get rid of, since they depend on the presence of cylinders.

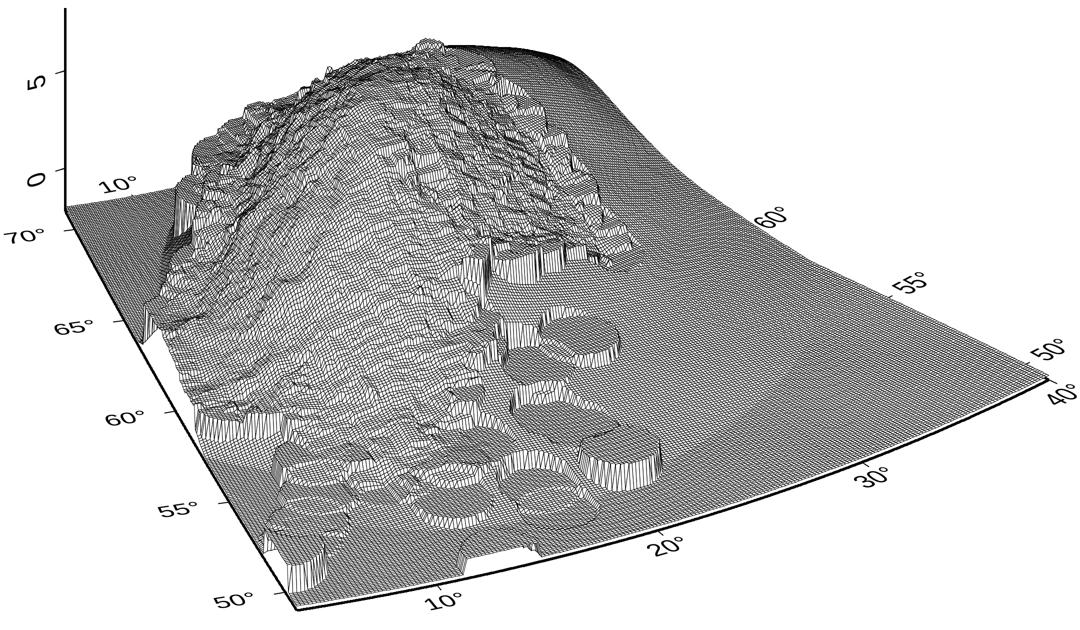

Figure 3.1: Vestøl's grid model extended with Lambeck's model in the areas where the former is undefined. Unit: mm/year.

One problem with the model in Fig. 3.1 is that the jumps are terribly abrupt, which makes the model ill-suited for practical application. Imagine the Baltic Levelling Ring passing through the cylinder at latitude 52 and longitude 20 degrees. The 2 mm/year jump will cause errors both over longer distances and locally. To diminish the latter (high-frequency) errors, it seems favourable to make a *gradual* passage from Vestøl's model to Lambeck's. Such a model is presented in Fig. 3.2. It was derived under the assumption that Vestøl's grid should not be changed inside its definition area. A smooth transition zone is then assumed to lead over to Lambeck. To accomplish this transition, the difference between Vestøl and Lambeck is interpolated using exact inverse distance interpolation as implemented in SURFER 8 (Golden Software 2002). The prediction points are chosen for the whole original grid, but the observation points are limited to the Vestøl definition area. After interpolation, Lambeck's model is restored. Since the interpolation is exact, Vestøl's original grid is reproduced exactly as required.

In inverse distance interpolation, sometimes also known as Bjerhammar's deterministic prediction method (see Bjerhammar 1973), the value in the prediction (grid) point i is determined as a weighted mean of the given observations, using the weight p_{ij} for observation j,

$$p_{ij} = \frac{1}{\left(\sqrt{d_{ij}^2 + s^2}\right)^p} \tag{3.1}$$

where d_{ij} is the distance between the prediction point i and the observation point j, s is the so called smoothing parameter and p is the weighting power (or power parameter). In case the smoothing parameter s is set to zero, the interpolation is *exact*, which means that the observations are reproduced exactly. For non-zero s, the given observations are filtered and the interpolator is *smoothing*. If the prediction point i is further removed from the observations j, the extrapolation approaches the mean value of all observations. The speed of this transition depends on the power p. For low values, the mean value is approached faster. This can easily be seen by expanding the distance according to the binomial theorem,

$$d_{ij}^p = D\left(1 + \frac{\Delta d_{ij}}{D}\right)^p \approx D\left(1 + p\frac{\Delta d_{ij}}{D}\right) \tag{3.2}$$

where D is the distance to the centre of all observations and Δd_{ij} is the residual distance. In case the linear term becomes so small that it may be neglected, an ordinary mean value is obtained, assuming for the moment that $s = 0$. When $p \to \infty$, on the other hand, the result approaches the nearest neighbour. All this means that for reasonably small powers, the extrapolation approaches the mean value of the difference between Vestøl and Lambeck far away, which is equal to -0.19 mm/year in the present case. This implies that far away from Vestøl's model, the new model approaches -2.19 mm/year instead of -2.00 mm/year. However, as the digitised version of Lambeck's model is defined rather arbitrarily using the minimum value in question (cf. the discussion in Section 2.3), it seems justified to remove the mean value so that a new minimum is obtained. The authors further believe that it is in order to adjust Lambeck's model up or down so that it fits all the available observations in the mean value sense. In the tuning of the model in Lambeck et al. (1998), the choice of the eustatic sea level rise using mareograph and other information accomplishes more or less the same thing. Thus, the remove-compute-restore interpolation is applied with respect to Lambeck's model with the mean value difference from Vestøl removed, which means that a smooth transition is obtained to the mean value shifted version of Lambeck's model. It should further be pointed out that the major reason for choosing inverse distance interpolation in the present case is that it provides a fast interpolation that makes it possible to utilise all observations for the interpolation of each grid point. Consequently, we are not forced to use some kind of search algorithm, in which a subset of the observations is picked out for each prediction. This is an advantage, since search algorithms tend to result in "staircase" behaviour in areas with only a few observations; cf. the cylinders in Vestøl's model.

The question now is how the power parameter p should be chosen. This can be determined empirically so that a nice looking transition between Vestøl and Lambeck is obtained. After trying several values for p, it was decided graphically (by studying wireframe plots) that $p = 3$ is optimal. The resulting model is presented in Fig. 3.2. It is clear that the method works well and that a smooth passage is indeed obtained to Lambeck's mean value shifted model.

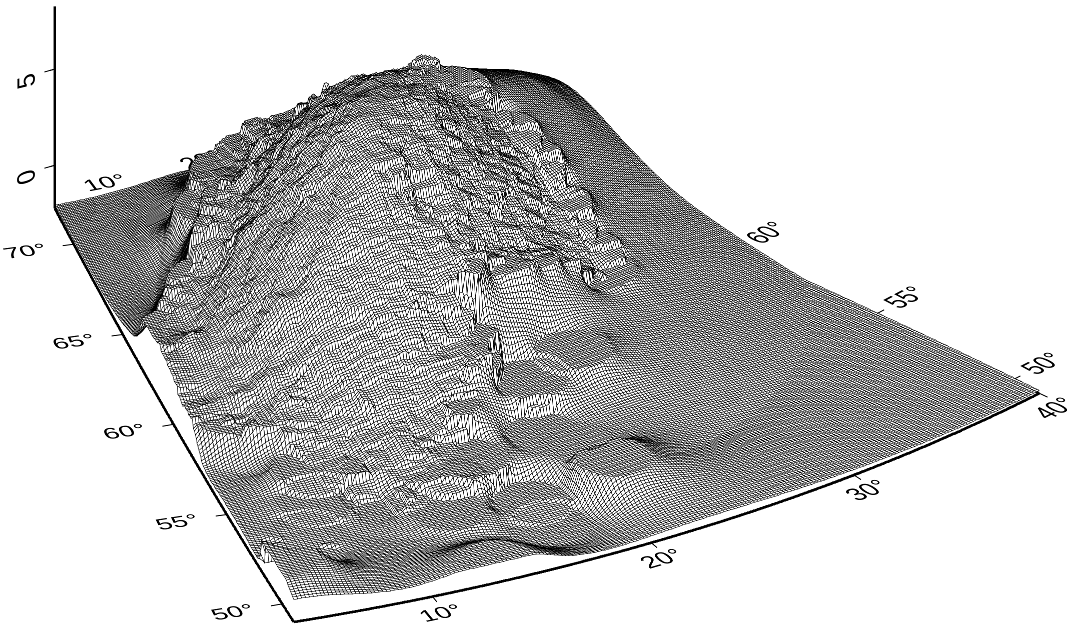

Figure 3.2: Vestøl's grid model continued with the Lambeck model (mean value shifted). Smooth transition using inverse distance interpolation (power 3 and no smoothing) of the difference between the two models. Unit: mm/year.

A similar model was produced by Karsten Engsager at KMS in Denmark for use within the working group for height determination (Engsager 2004, email). In this case, however, least squares collocation was utilised to accomplish the transition. The software used by Engsager is the GRAVSOFT program GEOGRID written by René Forsberg (Forsberg 2003). In this program, a second order Markov covariance function is assumed with a certain correlation length. It is possible to weight each observation using apriori standard errors, which yields a diagonal covariance matrix $\boldsymbol{D}$ in Eqs. (2.3) and (2.4). The variance C_0 of the covariance function is then determined directly from the observations. Engsager applied 120 km correlation length with low apriori standard errors for Vestøl's model (0.1 mm/year) and much higher for Lambeck (5.0 mm/year). One version of the model is presented in Fig. 3.3. Please notice that this model refers to a previous version of Vestøl's model, which does not consider the uplift dependent part between absolute and apparent uplift in Eq. (2.11). Since the presentation of this extension, Engsager has continued to work on new models, but the deadline that RH 2000 should be released the 1st of February 2005 forced us to focus on the problem at depth ourselves (as fast as possible). The second model of Engsager released in February 2005 is therefore not considered here. It seems to have been produced along similar lines with the only difference that the latest version of Vestøl’s model is used (the same one as is utilised in this report).

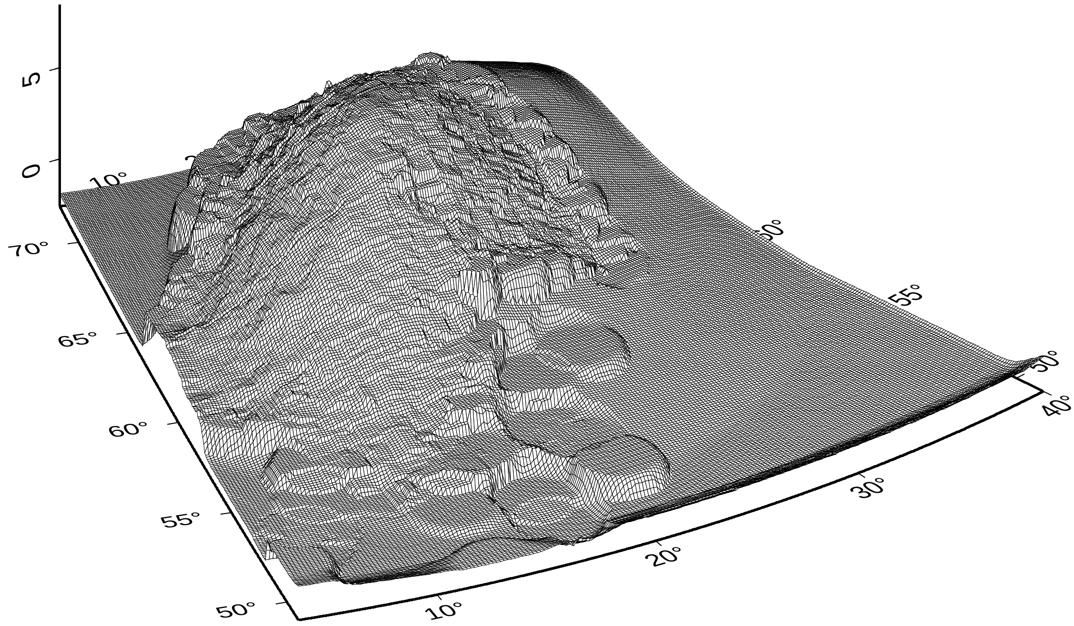

Figure 3.3: An earlier version of Vestøl's grid model extended by the Lambeck model using least squares collocation. The UV1 model (Engsager, email, 2004). Unit: mm/year.

Now, if the last two models are studied carefully, it can be concluded that the smooth transition only improves the situation marginally. The cylinders introduced by the gridding are still present, and the inconsistency of the GPS observations in continental Europe and to the southeast of the Baltic Sea is as disturbing as before. No matter how one handles or extends Vestøl’s grid model, it is simply impossible to get rid of the discrepancies between the bad GPS observations. Studying the estimated standard errors in Fig. 2.12 for the GPS stations in question, it can be seen that they are comparatively high. It might be asked whether this information is really of any use at all at the present time. Is it not better to prefer the geophysical model for the areas in question, at least in case the latter agrees reasonably with the observations in some kind of mean sense? This is the opinion of the authors. It was thus decided to neglect all the grid nodes below latitude 54° for longitudes smaller than 19° and below latitude 59° for larger longitudes. The dividing line is chosen so that it is situated outside the border of Sweden, so that the differences between Lambeck's and Vestøl's models are as small as possible along the line and so that the GPS-stations responsible for the cylinders are excluded. Of course, the most logical thing to do would be to recompute Vestøl's model without the bad observations, but the time limitations mentioned above forced us to take both Vestøl's and Lambeck's models in the shape they were at the beginning of 2005. To obtain a simple solution, it was decided to use the dividing line described above, which is illustrated in Figs. 2.1, 2.2, 2.5 and 2.9. Exactly the same inverse distance interpolation was applied as was used for the model presented in Fig. 3.2, but only observations above the dividing line were used. The resulting model is illustrated in Fig. 4.

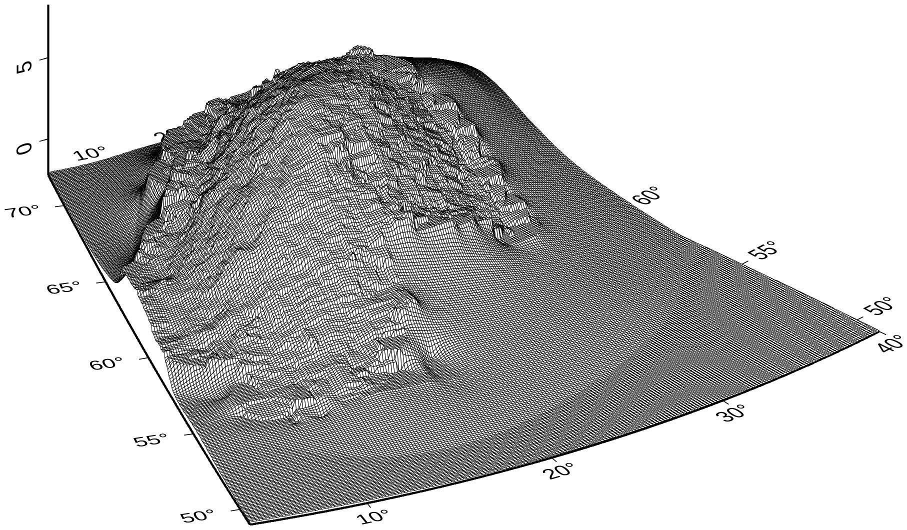

Figure 3.4: Vestøl's grid model above the dividing line extended with the Lambeck model. Smooth transition using inverse distance interpolation (power 3 and no smoothing). Unit: mm/year.

In this way, one gets rid of the largest cylinders to the south, but the plateaus are still present at the outskirts of the model. It is of course questionable how good the model is, and how well it agrees with the observations. In any case, the model in Fig. 3.4 looks more like real uplift than the ones in Figs. 3.1 to 3.3, but it might be argued that Vestøl's model is still too rough in the central parts of the area; cf. the "zigzag" contour lines in Fig. 2.11. What can be seen in the above figures is not pure land uplift, but a combination of land uplift, other geodynamic phenomena of continuous or discontinuous (tectonic) nature *and* observation errors. It is believed that it is mostly levelling errors that are responsible for the rough appearance.

One way to get rid of most of the remaining cylinders, at the same time as the grid looks more realistic, is to use a smoothing interpolator. The grid could of course also be smoothed separately. This type of model was computed using inverse distance interpolation with power $p = 3$ and the smoothing parameter $s = 0.5$ degrees. The values of the parameters were determined by trial and error so that a suitable model was obtained. This is discussed in more detail in the next section. The smoothed grid is presented in Fig. 3.5.

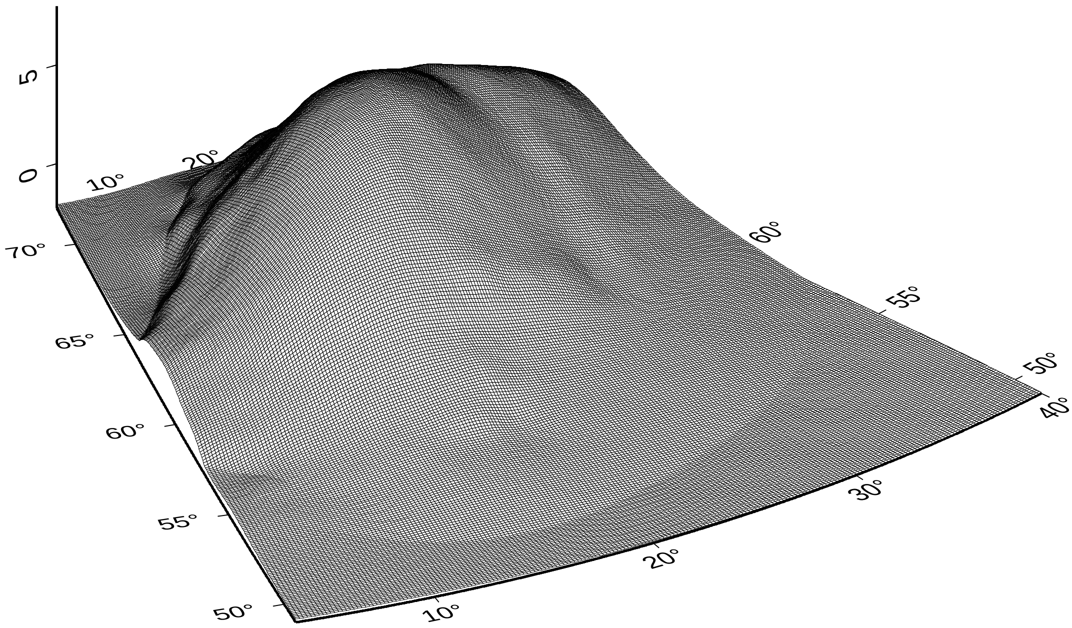

Figure 3.5: Vestøl's grid model above the dividing line extended with the Lambeck model. Smooth transition using inverse distance interpolation with power 3 and the smoothing parameter 0.5 degrees. Unit: mm/year.

As can be seen, the land uplift pattern now looks much more realistic. One might ask, though, whether the smoothing parameter has been chosen in an optimal way. Another important question is whether we really get rid of the cylinders at the limits of Vestøl's model. The last question is obviously important. At this point in our investigations it was discovered that Vestøl had applied an independent gridding algorithm, which explains some of the problematic features of the grid model like the cylinders. As was discussed at the end of subsection 2.2.1, it therefore seems like the best solution to go back to the adjusted point values, which is the topic of the next section. Another important question that has been posed by the above tests is how much the land uplift model should be smoothed. The more the model is smoothed, the more it looks like a geophysical model. Due to the rather high flexural rigidity of the lithosphere, it is simply impossible for the model to take “any” shape. However, the more the surface is smoothed, the less exact becomes the interpolation, i.e. the more low-pass filtered the observations become. In the next section it will therefore also be investigated exactly how close the interpolated surface is to the original observations for different amounts of smoothing.

3.2 Interpolation and extrapolation of Vestøl’s model as defined in the observation points

At the end of the last section it was concluded that the best option is to go back to the original point values, which were obtained by least squares collocation with unknown parameters from all the available observations. Throughout the rest of this report, it will be assumed that only the point values above the dividing line are used. There is simply no way to reconcile the GPS observations in the central parts of Europe. Unfortunately this line was chosen so that also 5 tide gauges were removed, but at least one of them deviates quite a lot from the neighbouring observations; see Fig. 2.1. This unintended exclusion is unfortunate, but there was no time to correct the mistake. Fortunately, it makes very little difference for the resulting uplift model. The omitted tide gauge uplifts are reproduced sufficiently well by the different models anyhow (see e.g. Fig. 4.4 below).

The question now is how the land uplift values should be interpolated and extrapolated from Vestøl’s point values above the dividing line (743 observations). Many methods exist with different properties, and it is not apriori clear which one that is most suitable for the present case. In the last section, standard inverse distance weighting according to Eq. (3.1) was applied for interpolation (smoothing) and/or extrapolation. This technique is also referred to as Bjerhammar’s deterministic method (see Bjerhammar 1973). Another possible alternative is to use least squares collocation (Moritz 1980; Forsberg 2003) or Kriging (Cressie 1991). One complication in the present case is that we want to extrapolate the difference from Lambeck’s model (mean value shifted) in such a way that the difference goes to zero after a certain distance. This makes several other interpolation methods unsuitable. For instance, minimum curvature methods produce a nice field in areas with observations, but the surface tends to behave arbitrarily where there is no information. Below, inverse distance interpolation will first be investigated, both with and without filtering of Vestøl's point observations. After that, a few versions of least squares collocation (Kriging) are considered and what is believed to be the most suitable method is chosen.

It should be mentioned that the same remove-compute-restore technique as above is applied. In the remove step, Lambeck's geophysical model is subtracted from the 743 point observations, where it is assumed that the latter model has been shifted so that the mean of the differences betweens the Vestøl observations and the Lambeck model is zero. As the 743 observation points are irregularly distributed, the mean value deviates to the grid mean above. The *point mean value* is -0.684 mm/year. After the difference has been gridded using the interpolation method in question, Lambeck's mean value shifted model is restored; cf. the discussion in the last section.

3.2.1 Exact inverse distance interpolation

The first method to be tested with point data is *exact* inverse distance interpolation. As discussed in the last section, this method approaches the mean value far away from the observation points. This makes it suitable in the present case, since this means that the final result will approach Lambeck exactly as required. The power parameter p was chosen to 3 after some experimenting with different values. The resulting model is illustrated by a wireframe plot in Fig. 3.6 and by contour lines in Fig. 3.7.

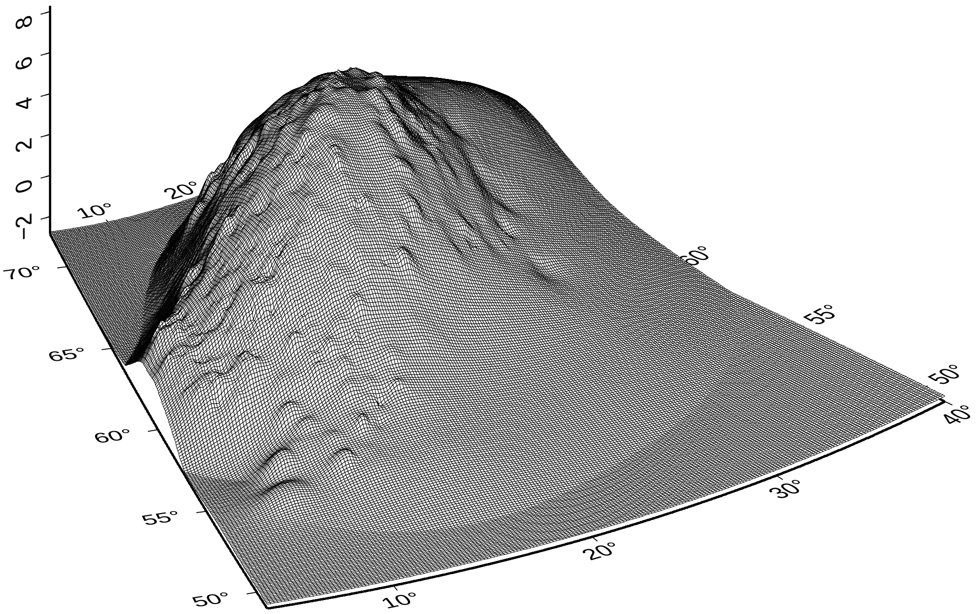

Figure 3.6: Vestøl's point model above the dividing line extended with the Lambeck model. Exact inverse distance interpolation/extrapolation (power 3, no smoothing). Unit: mm/year.

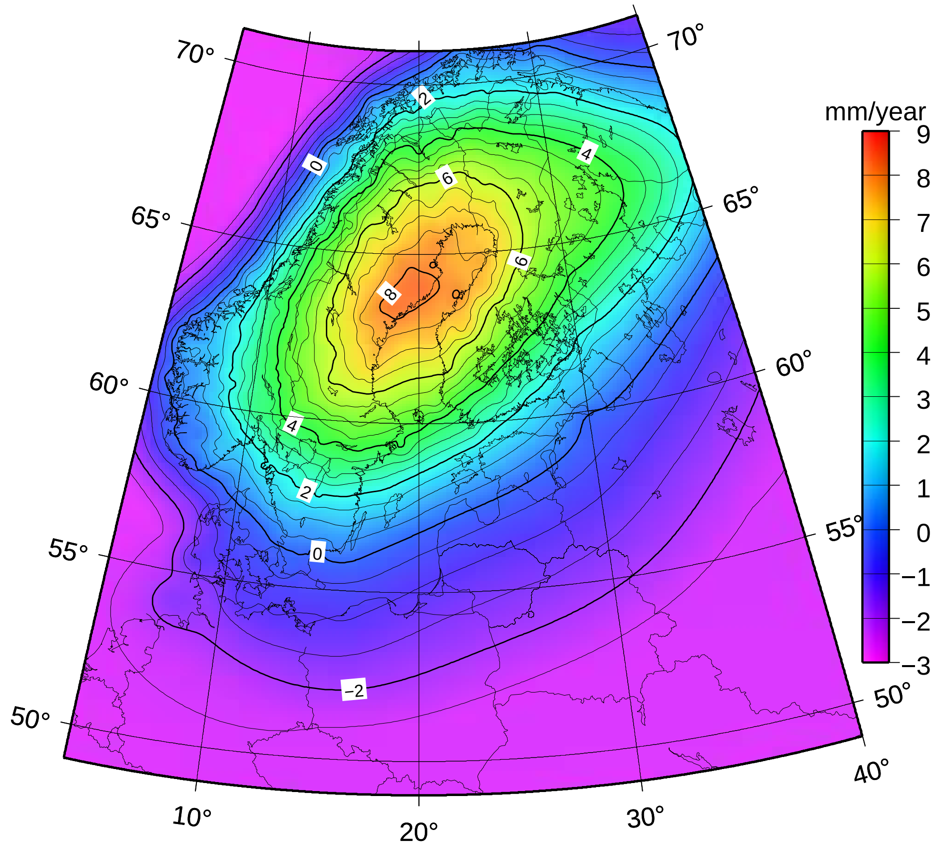

Figure 3.7: Contour lines for Vestøl's point model extended with the Lambeck model. Exact inverse distance interpolation/extrapolation (power 3, no smoothing). Unit: mm/year.

The first thing that can be seen in the above figures is that interpolation from the observation points produces a much better looking grid compared to Vestøl's original one in Figs. 2.10 and 2.11. The cylinders completely disappear and a gradual passage to the mean value shifted model of Lambeck is obtained. It is thus concluded that the uplift model should be interpolated from Vestøl's point values. Another thing that can be observed in Fig. 3.6 is the typical bull's eye patterns generated by exact inverse distance interpolation. It is a well-known feature of this type of interpolation that it produces small volcanoes around the observations in case they differ from the general trend of the surface (e.g. Golden software Inc. 2002). Furthermore, by studying Figs. 3.6 and 3.7, it seems clear that the grid is a little too rough to be realistic and that some smoothing might be called for.

However, before turning to the question of smoothing, let us take a closer look at how well the present model fits with Lambeck's counterpart and the given observations. The difference from Lambeck's model (not mean value shifted) is shown in Fig. 3.8. Notice that the remove-compute-restore method is applied with respect to the mean value reduced model, at the same time as the differences in Fig. 3.8 are presented with respect to Lambeck's model as it is; see Section 2.3. The statistics of the residuals with respect to the tide gauge and GPS data are then presented in Table 3.1

Table 3.1: Statistics for the apparent uplift residuals for Vestøl's point model interpolated using the *exact inverse distance* method. The maximum for "All tide gauges" is given for both the outlier stations Furuögrund/Oslo. Unit: mm/year.

Observations	#	Min	Max	Mean	StdDev	RMS
All tide gauges	58	-0.57	0.87/1.20	0.03	0.21	0.21
Edited tide gauges	56	-0.47	0.35	0.0	0.09	0.09
All GPS	55	-1.25	2.07	0.12	0.61	0.62
SWEPOS GPS	21	-0.48	0.33	-0.03	0.22	0.22

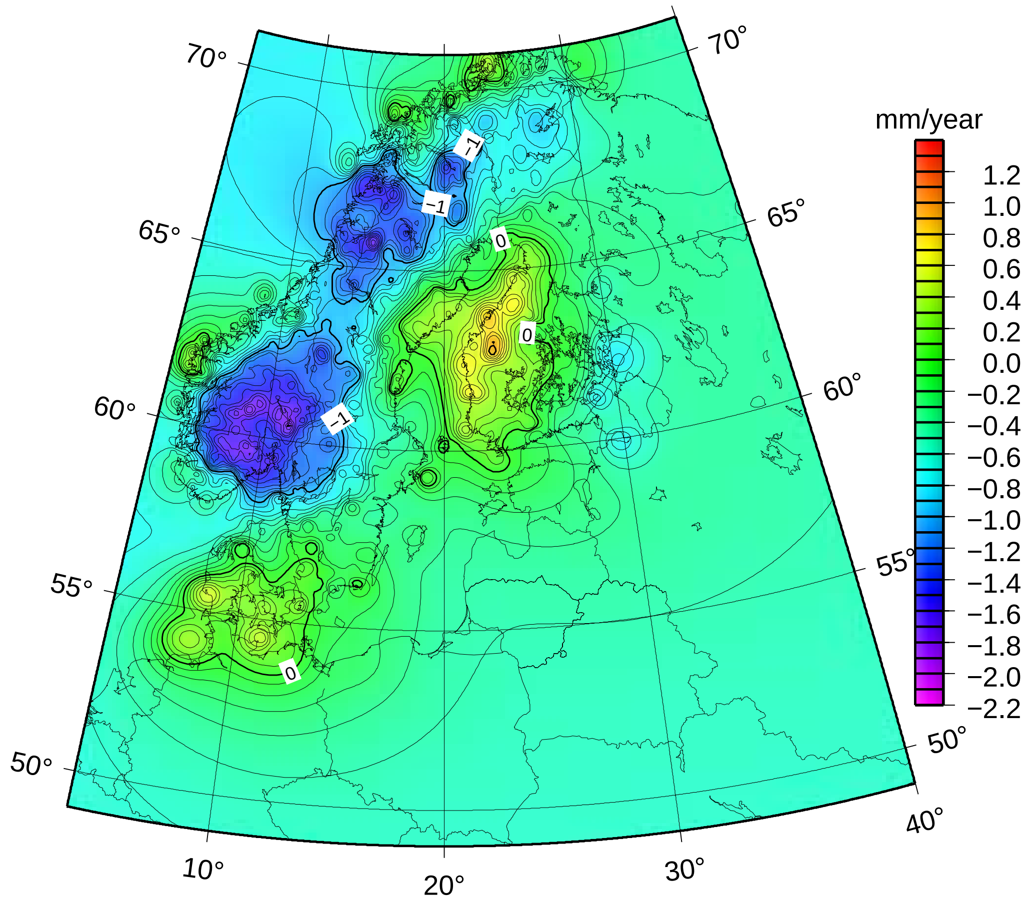

Figure 3.8: Contour lines for the difference between the *exact inverse distance* and the *Lambeck* models. Unit: mm/year.

By studying Fig. 3.8 it can be seen that Lambeck's model is significantly improved. The largest absolute improvement is 2.2 mm/year and occurs in the southern parts of the Norwegian mountains. Furthermore, the difference is small along the Swedish coast, where there are many tide gauges, which is only what could be expected considering the way the model was constructed; see Chapter 1 and Section 2.3. For some reason Lambeck's model deviates considerably at the mareographs on the Finnish side of the Gulf of Bothnia. Furthermore, the characteristic bull's eye patterns can now be seen also in the contour lines. Notice for instance the deviating uplifts along the coast of Norway. Several of these occur in the open levelling lines discussed at the end of Subsection 2.2.1. The statistics in Table 3.1 contain no surprises, but the residuals are somewhat larger compared to Vestøl's grid model in Table 2.3. This mainly depends on only point values above the dividing line being utilised to interpolate the model in Table 3.1. On average, however, the statistics in Tables 2.3 and 3.1 agree well.

To sum up, it is clear that we should start with Vestøl's point values. It has further been found that exact inverse distance interpolation is not optimal for the task, mainly due to the bull's eye patterns and to the rough appearance of the resulting grid. The former should obviously not be present in the final model, but the phenomenon is nevertheless helpful in spotting deviating observations. In the same way as above, it might be argued that it is suitable to smooth the model, but now this need not be motivated by the reduction of the disturbing cylinders. Instead, the physics of the Earth dictates that the land uplift cannot have any arbitrary shape. What lies behind the deviating observations are therefore mainly observation errors, most likely in the levelling lines. To reduce the influence of the errors, it is suitable to smooth the model. "When in doubt, smooth" (Moritz 1980).

3.2.2 Smoothing inverse distance interpolation

The philosophy now is thus to produce a grid using a smoothing inverse distance interpolation, which means that the given observations will be filtered. The main question at this point is how to choose the smoothing parameter s in Eq. (3.1). The smoothing parameter s can be interpreted using the following fact. If smoothing inverse distance interpolation is used with a certain value for s, very similar results are obtained compared to first using exact inverse distance interpolation followed by applying a moving average filter that averages over a circle with radius s. This should be viewed as an empirical statement that has been corroborated by numerical tests of the authors. It is perhaps not valid for all powers p, but it is a good approximation for the powers in question here, i.e. $p = 3$. Thus, what can be expected from the smoothing inverse distance interpolation for different values of s might be visualised as the corresponding moving average (using the same radius s) of the result from exact inverse distance interpolation. Now, as there are no apriori reasons for preferring a certain value, s was chosen empirically using the following criteria, which are applied in the rest of this report to find the most suitable interpolation for RH 2000. The resulting model should

1. "look" realistic, which implies that it should be reasonably smooth. It is true that the postglacial uplift might have a tectonic component, which occurs on a more local scale. However, from the analysis of very long GPS time series, it seems like tectonic movements are rare and/or small; cf. Johansson (2002) and Lidberg (2004). Even in case such a component is significant, it is unlikely that we will be able to model it satisfactorily. What we are doing here is simply to try to find a model for the continuous part of the uplift, which is bound to be smooth.

2. fit reasonably with the observations. In our case, the GPS and tide gauge data are used to study this aspect. As was mentioned above, the residuals in the levelling observations are more difficult to summarise and visualise. The standard errors can be used to judge how much the model can be allowed to differ from the observations. It seems suitable to use the standard errors 0.2 mm/year for the tide gauges (Ekman 1996) and 0.3 mm/year for the SWEPOS stations (Lidberg 2004). In case all GPS observations are considered, it should be remembered that the bad point observations south of the dividing line have been discarded.

3. behave well in areas without observations. This means that the model should not only be realistic for the *whole* Baltic Levelling Ring but also in other areas without observations, for instance in the Baltic Sea. Even though the main purpose of this work is to construct a land uplift model for RH 2000, it is always possible that the model is used in other areas for other applications in the future. This criterion is primarily intended to exclude interpolation methods that behave more or less arbitrarily outside the observations.

The last item is obviously not problematic for inverse distance interpolation. As can be seen from the exact case above, the method performs well in areas without observations, both inside and outside (extrapolation) the observation area. Of course, the corresponding lack of information means that the uplift cannot be expected to be especially accurate, but the model does not start to behave violently or oscillate with large amounts. The value of the smoothing parameter s was thus chosen empirically balancing criterion 1 and 2 against each other. Otherwise exactly the same remove-compute-restore technique as above was used and the power parameter p was chosen to 3. The best attempt, which was found for s = 0.5 degrees, is presented in Figs. 3.9 and 3.10. The difference from Lambeck is illustrated in Fig. 3.11. As can be seen, the resulting model is considerably smoother than the exact counterpart in Figs. 3.6 to 3.8. According to the authors it “looks” more realistic. It resembles Lambeck’s original model in Figs. 2.14 and 2.15, but it is important to notice that the smooth inverse distance model actually differs considerably from Lambeck. The difference, which can be found in Fig. 3.11, speaks for itself. However, that the model looks nice is of no use in case it does not fit with reality. The statistics for the comparison with the tide gauge and GPS observations are given in Table 3.2, which also contain statistics for the difference to Vestøl’s point values above the dividing line.

Table 3.2: Statistics for the apparent uplift residuals for Vestøl’s point model interpolated using the *smoothing inverse distance* method. The maximum for “All tide gauges” is given for both the outlier stations Furuögrund/Oslo. Unit: mm/year.

Observations	#	Min	Max	Mean	StdDev	RMS
Vestøl’s point values above the dividing line	743	-0.65	0.65	0.02	0.20	0.20
All tide gauges	58	-0.36	0.93/1.24	0.18	0.25	0.31
Edited tide gauges	56	-0.36	0.55	0.15	0.19	0.24
All GPS	55	-1.15	2.07	0.18	0.61	0.64
SWEPOS GPS	21	-0.59	0.49	0.03	0.32	0.32

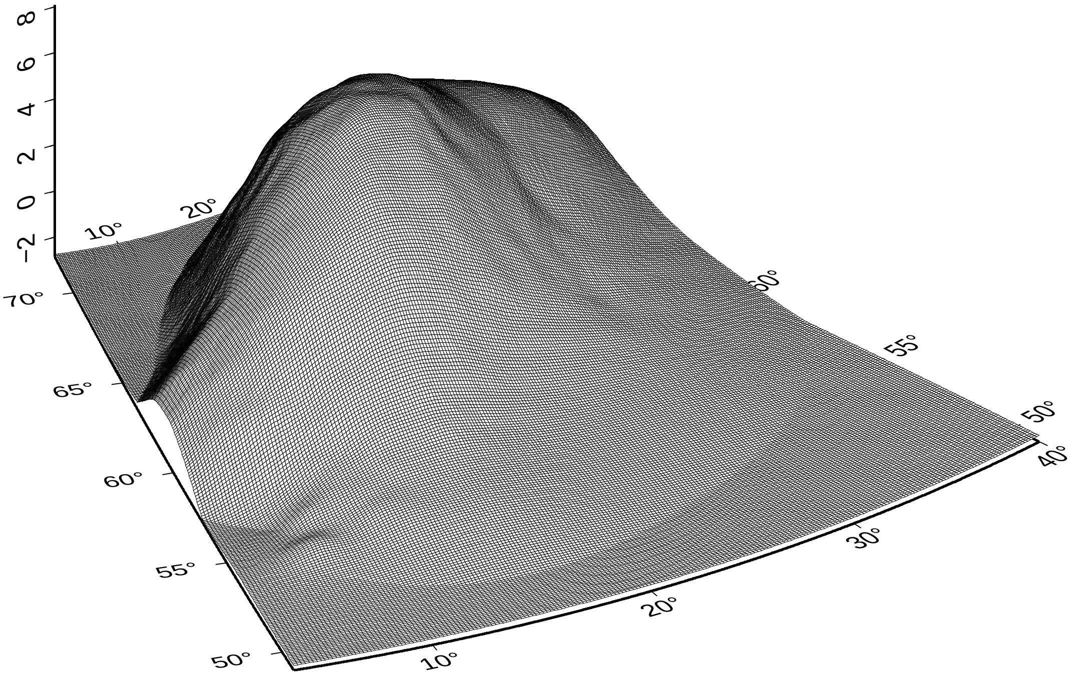

Figure 3.9: Vestøl's point model above the dividing line extended with Lambeck. Smoothing inverse distance interpolation/extrapolation (power 3, 0.5 degree smoothing parameter). Unit: mm/year.

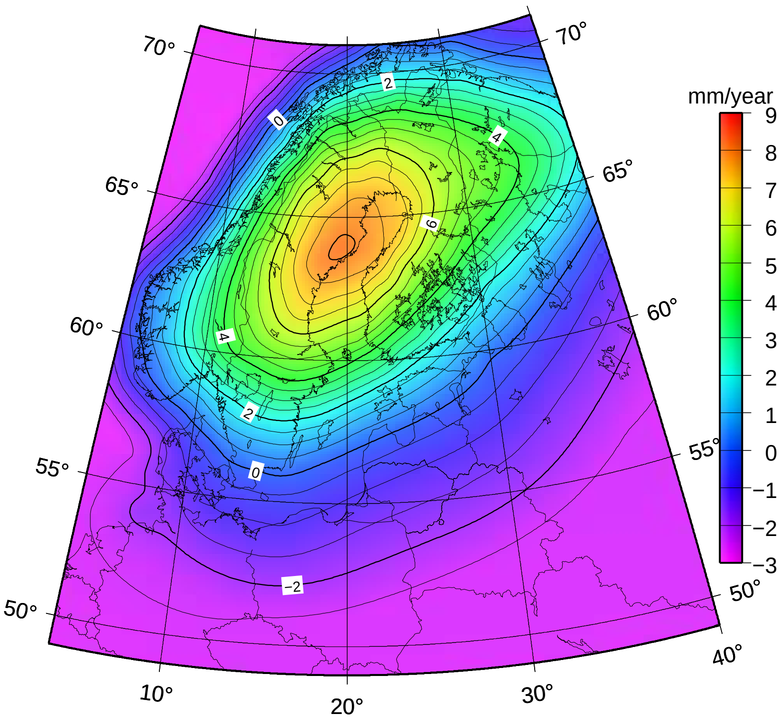

Figure 3.10: Contour lines for Vestøl's point model extended with Lambeck. Smoothing inverse distance interpolation/extrapolation (power 3, 0.5 degree smoothing parameter). Unit: mm/year.

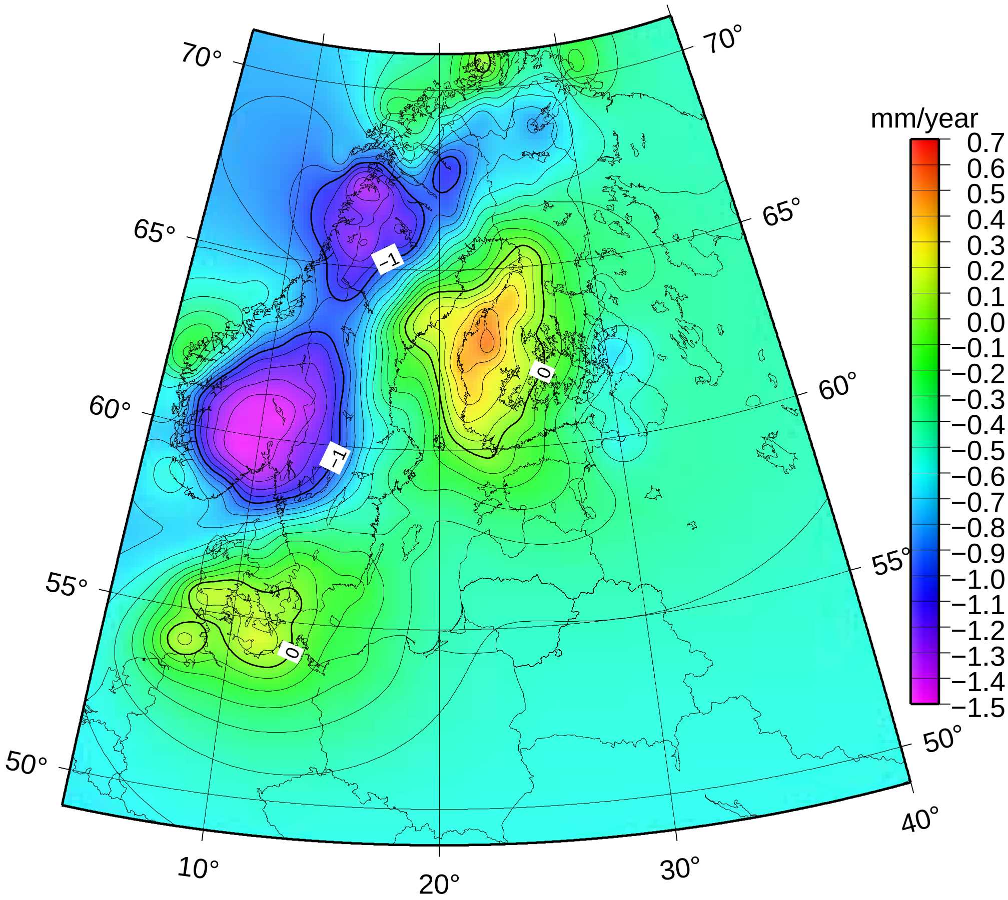

Figure 3.11: Contour lines for the difference between the *smoothing inverse distance* and *Lambeck* models. Unit: mm/year.

Let us first consider the fit to the tide gauges. Disregarding the two outliers discussed in Subsection 2.2.2, the RMS of the edited residuals is 0.24 mm/year, which is perhaps a bit high, but nevertheless acceptable considering the standard error 0.2 mm/year in Ekman (1996). As is clear from our discussion of variance component estimation in Subsection 2.2.2, we are sceptical concerning Vestøl's contention that the tide gauge standard errors are as low as 0.1 mm/year for the whole region.

Another observation that can be made in Table 3.2 is that the model seems biased at the tide gauges: The mean value 0.15 mm/year is significantly different from zero. This feature depends on another property of smoothing inverse distance interpolation. Remember that this type of interpolation yields similar results to first using the exact inverse distance method and then applying a moving average filter. It is a well-known feature of moving average filtering that it is biased by a positive amount in areas with a positive second derivative (convex upwards) and vice versa (Press et al. 1992, p. 645). Since it is the difference between the exact inverse distance and the Lambeck models that is interpolated (see Fig. 3.8), it follows that the smoothed model will be too high in the "convex upward" parts in Norway and too low in the "convex downward" parts in Finland and Denmark. It is obvious that the moving average yields too high values in case the centre of curvature of the general trend is situated above the surface and vice versa. Now, as the majority of tide gauges are situated in the "convex downwards" areas (cf. Figs. 2.1 and 3.8), it follows that a systematically positive mean value can be expected. This is clearly a negative property of smoothing inverse distance interpolation, which needs to be considered when the final interpolation method is chosen. However, in the present case it is not certain that the bias is only negative. Since the largest deviations show up in the southern Norwegian mountains, where large differences from Lambeck's model have been derived using exclusively non-repeated levelling, it might be good that the interpolation damps the uplift values somewhat. This question will not be further discussed at the present point. It will be touched upon again when the choice of final interpolation method is discussed in Section 3.2.4.

It can further be seen in Table 3.2 that the agreement is acceptable for the SWEPOS stations and that the RMS value is approximately twice as high in case all the GPS stations are considered. It is concluded that the model agrees sufficiently well with the given observations, considering the standard errors in Section 2.1.

Consider now the difference from Vestøl's original model in Table 3.2. The RMS value is 0.2 mm/year with the extremes as large as 0.65 mm/year. This might seem too much, but it should be remembered that we actually want to smooth the data more than Vestøl (2005). This has been motivated several times above. The difference between using the exact and smoothing interpolation methods is illustrated in Fig. 3.12, in which Vestøl's point values have also been marked. As can be seen, a considerable amount of high-frequency variations have been filtered out. Another feature in Fig. 3.12 is that the difference is slightly negative on an average in the southern Norwegian mountains and slightly positive in the western parts of Finland, which is what could be expected according to the above reasoning that the moving average is too high in "convex upwards" areas and vice versa.

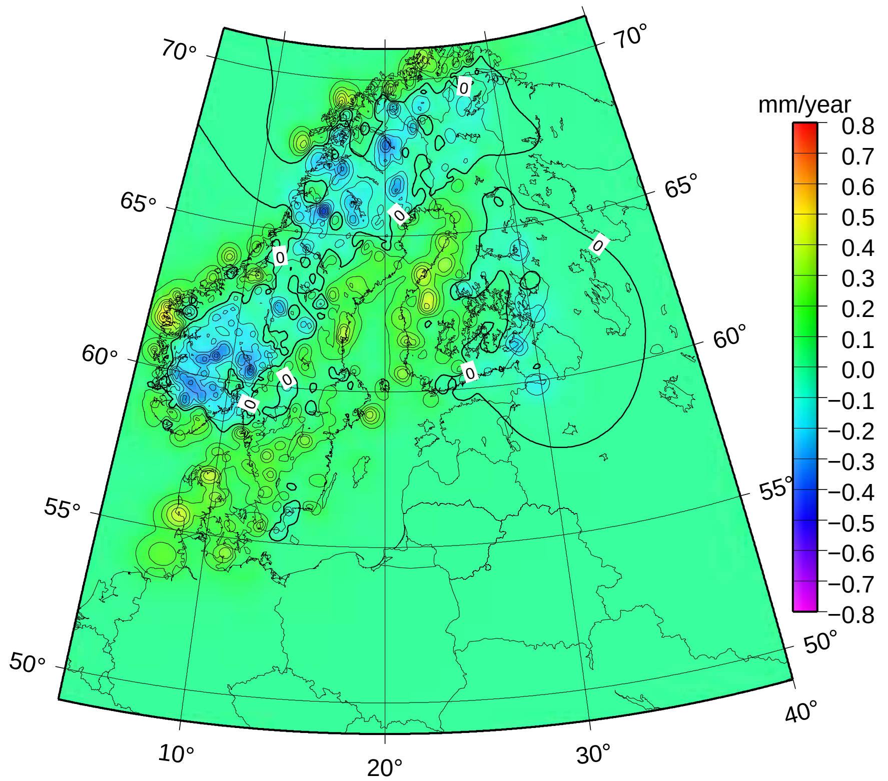

Figure 3.12: Contour lines for the difference between the *exact* and *smoothing inverse distance* interpolation models. The dots denote Vestøl's point values. Unit: mm/year.

Now, it might be objected that too much information has been thrown away by the smoothing, or that the amount of smoothing is not suitable. Above, the method has been to choose the smoothing parameter as high as possible under the constraint that the resulting model should not deviate more in the RMS sense from the given observations than their standard errors. It is believed that this procedure is preferable since it will reduce the influence of levelling and other errors. It is of course difficult to say with certainty that we have chosen exactly the correct degree of smoothing or that we have not filtered away too much of the signal buried in the observations. In order to see how important this choice is for the final model, the sensitivity of the estimated heights on the degree of smoothing was investigated. It was thus tested how dependent the final heights are on the choice of s by computing the whole Baltic Levelling Ring with both the exact and smoothing inverse distance interpolation models. The difference between the two models is illustrated in Figs. 3.13 and 3.14 for the whole area and Sweden, respectively. Some statistics can be found in Table 3.3. To visualise the difference between the adjusted levelling heights, Delaunay triangulation with linear interpolation inside each triangle was used, which explains the values at sea in Fig. 3.13.

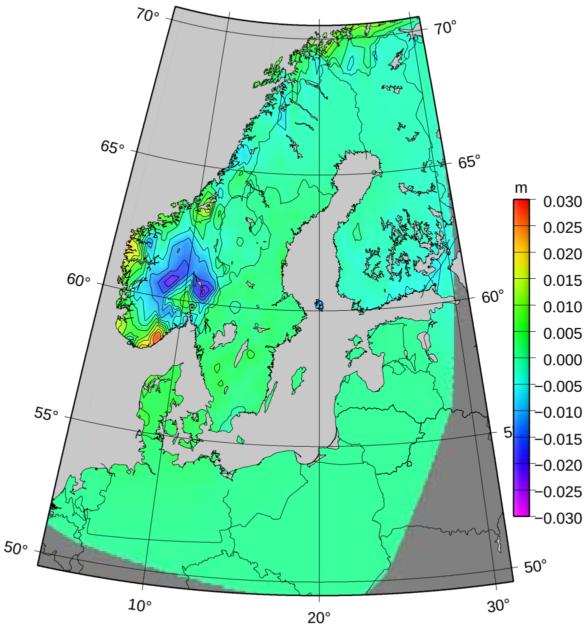

Figure 3.13: Adjusted height differences between using the *exact* and *smoothing inverse distance* interpolation models for the Baltic Levelling Ring. Delaunay triangulation with linear interpolation used for the visualisation. Unit: m.

Table 3.3: Statistics for the difference in adjusted heights between using the exact and smoothing inverse distance uplift models Unit: m.

Observations	#	Min	Max	Mean	StdDev	RMS
BLR	7401	-0.0287	0.0302	0.0006	0.0027	0.0027
RH 2000	5088	-0.0048	0.0068	0.0010	0.0017	0.0020

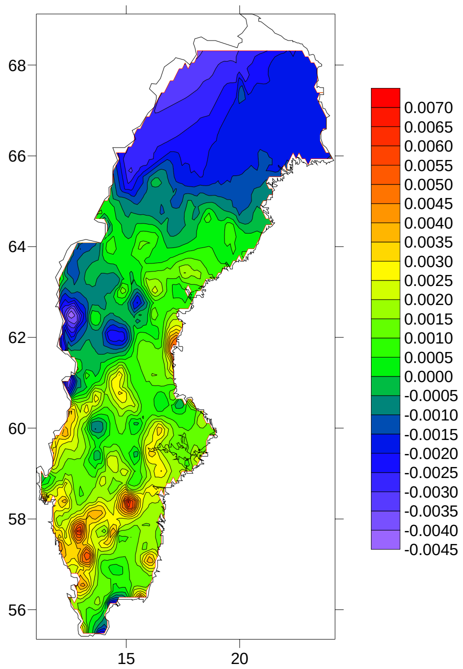

Figure 3.14: Adjusted height differences between using the *exact* and *smoothing inverse distance* interpolation models for the RH 2000 network in Sweden. Delaunay triangulation with linear interpolation used for the visualisation. Unit: m.

It can be seen that the dependence on the degree of smoothing is largest in the Norwegian areas with only non-repeated levelling. In addition, many old Norwegian lines are utilised in the height adjustment, which implies that the sensitivity in question is higher there. Fortunately, the situation is more promising otherwise. In Sweden, the third precise levelling started 1979 in the southern parts of the country and ended 1999 in the north (disregarding re-levelling). Since the reference epoch in the height adjustment is 2000.0, the land uplift corrections are comparatively small. However, the difference between exact and smoothing inverse distance interpolations is significant also in Sweden. Relative errors on the 5-10 mm level are introduced over short distances in the southern half of the country. Consider for instance the “volcano” in Östergötland ($\phi \approx 58.3^\circ$ and $\lambda \approx 15.1^\circ$). The only land uplift observations here are the second and third precise levellings, separated by approximately 30 years, which yields the significant uplift difference illustrated in Fig. 3.12. It seems likely that there is an undiscovered gross-error in the second levelling. Many other examples can be found of large local variations that can be attributed to levelling errors. Since the reliability for the third levelling is much higher than for the first and second counterparts, the gross errors are likely to be in the latter.

Another thing that can be observed in Fig. 3.14 is that the systematic differences caused by the moving average properties of the smoothing inverse distance interpolation (see the discussion above) yield significant effects for the adjusted heights. In this case, however, the effect is of a long-wavelength nature, which makes it rather harmless; cf. for instance the blue-purple area in the north of Sweden.

Thus, the use of an exact or a smoothing interpolation yields significant differences for the adjusted heights in the south of Sweden. The fact that the differences are lower in the north mainly depends on the corresponding lines being observed closer to the reference epoch 2000. To the authors, it seems best to trust the smoothing alternative. In the vast majority of cases, the difference between the exact and smoothing techniques can be blamed on land uplift errors caused by non-random behaviour of old levellings. If someone claims that we throw away important local information concerning the uplift field, for instance in Östergötland, then our stance is simply that we believe that it is more likely that the local effects are caused by undiscovered errors in the long, comparatively uncontrolled, lines of the first or second precise levellings. On the other hand, we still believe that levelling can provide useful information, which also motivates the present strategy; cf. the final discussion in Chapter 5. However, it is still questionable whether inverse distance interpolation is the most suitable method. In the next section a good alternative is investigated, namely least squares collocation or Kriging.

3.2.3 Kriging and least squares collocation

The main purpose of this subsection is to investigate least squares collocation and Kriging as alternatives to inverse distance interpolation. The two names refer to more or less the same thing, but are used in different contexts. The technique is denoted least squares collocation in Geodesy (e.g. Moritz 1980), while it is known as Kriging in Geostatistics (e.g. Cressie 1990). The main differences are that the terminology differs and that the covariance function in collocation is replaced by the (semi-) variogram in Kriging. The latter is defined as half the variance for the difference at two locations separated by the distance d.

Consider first least squares collocation, which was presented in a more general setting in Subsection 2.2.1. We now focus on the special case interpolation/extrapolation of a function in two-dimensional space. If it is assumed that the same remove-compute-restore technique as above is used, the prediction equation (2.5) may be written as

$$\hat{l}_P = \boldsymbol{C}_{l_P l}\left(\boldsymbol{C}_{ll} + \boldsymbol{D}\right)^{-1}\left(\boldsymbol{l} - \boldsymbol{l}_{lamb} - \overline{\left(\boldsymbol{l} - \boldsymbol{l}_{lamb}\right)}\right) + l_{lamb,P} + \left|\overline{\left(\boldsymbol{l} - \boldsymbol{l}_{lamb}\right)}\right| \qquad (3.3)$$

where $\boldsymbol{l}$ is the land uplift observation vector, $\boldsymbol{l}_{lamb}$ contains the uplift from Lambeck's model, $\hat{l}_P$ is the predicted uplift in P, $\boldsymbol{C}_{ll}$ is the signal covariance matrix for the spatially correlated land uplift differences in the observation points, $\boldsymbol{D}$ is the covariance matrix for random observation errors and $\boldsymbol{C}_{l_P l}$ is a vector of covariances between P and the observations. Furthermore, $\overline{\left(\boldsymbol{l} - \boldsymbol{l}_{lamb}\right)}$ denotes a column vector with the mean of the differences from Lambeck in the observation points. Eq. (3.3) is the basic collocation equation that will be utilised to estimate land uplift from Vestøl's *point values above the dividing line*. It can be shown that under the assumptions made (see Subsection 2.2.1), Eq. (3.3) is the Best Linear Unbiased Estimator (BLUE), i.e. the unbiased estimator with minimum variance (e.g. Moritz 1980). Notice further that it is assumed that the mean of the collocation argument ($\boldsymbol{l} - \boldsymbol{l}_{lamb} - \overline{\left(\boldsymbol{l} - \boldsymbol{l}_{lamb}\right)}$) is zero.

As mentioned above, the main difference between Kriging and least squares collocation is that the variogram is used to describe the statistical properties of the field to be interpolated/extrapolated. Let us now consider Kriging in more detail. It is first assumed that the land uplift is estimated as a linear combination of the available observations. Since it is not required in Kriging that the average of the observations (difference from Lambeck) is zero, the remove-compute-restore technique may be simplified to

$$\hat{l}_P = \sum_{i=1}^{n} w_i\left(l_i - l_{lamb,i}\right) + l_{lamb,p} \qquad (3.4)$$

where the index i runs over all the n observations. It is assumed that the data contains no trend, i.e. $E\left\{l_P - l_{P+d}\right\} = 0$, where l_{P+d} is the land uplift at the distance d from P (in any direction). Furthermore, the covariance has to be homogeneous and isotropic. The Kriging estimator is now derived so that it is the Best Linear Unbiased Estimator (BLUE). To be unbiased, it is elementary to show that the weighting coefficients must satisfy the condition

$$\sum_{i=1}^{n} w_i = 1. \tag{3.5}$$

It is furthermore easy to show that the requirement of minimum variance is satisfied in case the following n equations are fulfilled (Cressie 1991):

$$\sum_{i=1}^{n} w_i \gamma_{i,j} + \phi = \gamma_{j,P} \qquad \text{for all } j \tag{3.6}$$

where ϕ is a Lagrangian multiplier, $\gamma_{i,j}$ is the variogram for the variation between the observations i and j while $\gamma_{j,P}$ is the same quantity between observation j and the prediction point P.

It can thus be seen that least squares collocation and Kriging can be derived using the same BLUE criterion. As stated above, the most basic difference is that the variogram is used instead of the covariance function. In addition, the only way to consider random errors in Kriging is to modify the variogram by the addition of a so-called *Nugget effect*. Let us elaborate a little on this point. The variogram is defined as *half the variance of the difference* of two observations separated by the distance d. If it is assumed that the random errors of different observations are constant and uncorrelated, i.e. that the covariance matrix of the observations looks like

$$\boldsymbol{D} = \begin{bmatrix} \sigma_l^2 & 0 & & 0 \\ 0 & \sigma_l^2 & & \\ & & \ddots & \\ 0 & & & \sigma_l^2 \end{bmatrix}, \tag{3.6}$$

it follows from the usual definition of the variance and covariance that

$$\begin{aligned} E\left\{(l_P-\mu)(l_{P+d}-\mu)\right\} &= C(d) \\ E\left\{(l_P-\mu)^2\right\} &= C_0 + \sigma_l^2 \end{aligned} \quad . \tag{3.7}$$

where μ is the expected value of l_P. It can now easily be seen that the variogram $\gamma(d)$ is related to the spatially correlated covariance function $C(d)$ by

$$\begin{aligned} \gamma(d) &= \frac{E\left\{\left(l_P - l_{P+d}\right)^2\right\}}{2} = C_0 + \sigma_l^2 - C(d) \qquad && d > 0 \\ \gamma(d) &= \frac{E\left\{\left(l_P - l_{P+0}\right)^2\right\}}{2} = 0 && d = 0 \end{aligned} \tag{3.8}$$

where $C_0 = C(0)$ is the variance. Sometimes the variogram is defined so that the first row in Eq. (3.4) is applied also for $d = 0$, which means that $\gamma(0) = \sigma_l^2$. This seems to be the case with Kriging as implemented in SURFER 8 (Golden Software Inc. 2002), which is the Kriging software used in this study, although it also has possibilities to apply the convention in Eq. (3.8). The variance σ_l^2 is usually referred to as the Nugget effect. Since the variogram is assumed homogeneous, it cannot take observations of different quality into account. This follows from *all* statistic properties in Kriging being specified by means of the variogram, while the covariance in collocation is used to describe only the spatially correlated field at the same time as the observation errors are modelled by the *covariance matrix* ***D***. Another difference is that the methods behave differently far away from the observations, which is already indicated by Eqs. (3.3) and (3.4). If it is assumed that the covariance function approaches zero when the distance d increases, it follows from Eq. (3.3) that the collocation contribution to the solution vanishes. This will be the case also when the observations are biased with a mean value far from zero. In Kriging, on the other hand, condition (3.5) implies that the solution becomes a weighted mean of the observations. This means that the Kriging part of the solution in Eq. (3.4) approaches the mean far away, at least to the extent that numerical effects are disregarded (see below). This difference between collocation and Kriging is important in the present study as we are interested in extrapolation of the land uplift difference with respect to Lambeck's model far from the observations.

The application of collocation thus requires that the covariance function and the observation noise covariance matrix ***D*** are specified, while a variogram is needed in the Kriging case. Usually, analytical functions are used to construct the covariance or the variogram, where the defining parameters are obtained empirically by analysing the given observations. Concerning the observation noise, collocation allows the specification of individual apriori standard errors. If correlations between different observations are known, they can easily be incorporated into $\boldsymbol{D}$. As mentioned above, Kriging in its present form is limited to the specification of *one* Nugget effect, common to all observations, even though this variogram might be constructed from several different components.

Let us now consider interpolation and extrapolation from Vestøl's point values. It should be remembered that the latter are the result of an application of the more general least squares collocation method described in Subsection 2.2.1. This means that the point values were estimated assuming the covariance function in Eq. (2.9), which describes the residual field after removal of a fifth degree trend polynomial. The uplifts consequently have a certain variability, which depends on the choice of both apriori standard errors and the covariance function. As has been discussed several times above, it is believed that Vestøl's choice yields a field that is a little too rough. This means that it might be preferable to choose a longer correlation length than 25 km, which was originally used by Vestøl (2005). It must further be considered that Vestøl's covariance function (2.10) refers to the difference from a polynomial trend function, while the deviation from Lambeck is interpolated in the present case. This implies that a direct correspondence cannot be expected between the two covariance functions. It seems like the best choice in the present case is to derive the variogram or covariance function from an empirical analysis of the available point value residuals above the dividing lines, which constitute 743 point value observations (differences from Lambeck). In order not to delve into too much detail, only the results from these exercises are presented here (see below). Concerning the specification of apriori standard errors for the observation noise, it should be noticed that the estimated standard errors are also available for Vestøl's point values; cf. Subsection 2.2.1 and Fig. 2.12. Thus, if least squares collocation is used, the estimated standard errors may preferably be applied to construct the noise matrix $\boldsymbol{D}$. In Kriging we are forced to apply a common Nugget effect, which could be constructed using the variance 0.2 mm/year, or perhaps a bit higher.

Below the results using Kriging as implemented in SURFER 8 (Golden Software Inc. 2002) are first presented. After that, least squares collocation is tested using the GRAVSOFT program GEOGRID (Forsberg 2003). It is assumed that the remove-compute-restore techniques in either Eq. (3.3) or Eq. (3.4) is utilised. It might be thought that it is a good alternative to construct an exact Kriging interpolator by neglecting the Nugget effect. However, this might easily yield extremely bad results in areas without observations when the observations contain more high-frequency power compared to what is implied by the variogram. This is clearly illustrated by the following test, in which a Gaussian variogram without Nugget is assumed,

$$\gamma(h) = C\left(1 - \exp\left\{-h^2\right\}\right). \tag{3.9}$$

This variogram is formulated in terms of the normalised distance *h*, which is computed approximately as

$$h \approx \sqrt{\frac{\cos^2\bar{\phi} \cdot \Delta\lambda^2}{A^2} + \frac{\Delta\phi^2}{A^2}} \approx \sqrt{\frac{\Delta\lambda^2}{4A^2} + \frac{\Delta\phi^2}{A^2}} \tag{3.10}$$

with the parameters C = 0.4 mm^2/year^2 (scale), A = 1.5 degrees (range) and $\bar{\phi}$ is the mean latitude of the area. As can be seen from the last part of Eq. (3.10), the mean latitude is taken as 60 degrees in the present project. The corresponding covariance function has the approximate correlation length 170 km. The *correlation length* is here defined as the length for which the covariance is *half* the variance (Moritz 1980). The above parameters are obtained by empirical analysis of the Vestøl's point value differences from Lambeck's model. It should also be mentioned that all observations are utilised for the prediction of each grid point, which means that no quadrant search algorithm is used to speed up the computations. Now, the resulting solution is referred to as the *exact Kriging* (SURFER 8) model. Its difference from Lambeck is presented in Fig. 3.15.

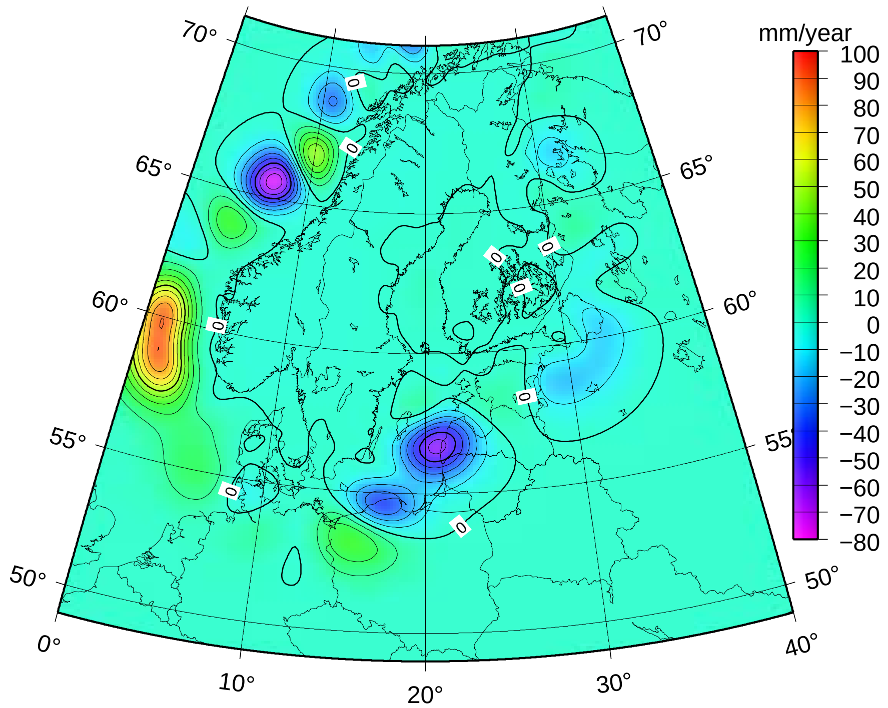

Figure 3.15: Contour lines for the difference between the *exact Kriging* and *Lambeck* models. Unit: mm/year.

The model fits perfectly with the observations (above the dividing line), but has the inconvenient feature that it starts to grow violently in unobserved areas. Notice that the unit is mm/year! It should be mentioned that this effect occurs also on more local scales. Thus, in case Kriging or collocation be used, it is more or less mandatory to choose a smoothing interpolator that considers the standard errors in the observations. It should be pointed out that this behaviour also depends on the shape of the variogram. It is, for instance, larger for a Gaussian variogram compared to a linear one. That the solution might behave in this way in exact Kriging is well worth to keep in mind. A similar behaviour is likely to show up also in case too small apriori standard errors are specified for the observations, but of course in a less exaggerated form.

Thus, we are more or less forced to choose a smoothing Kriging interpolation, i.e. Kriging with a Nugget effect. The solution using the same Gaussian variogram as above with a Nugget effect specified by the standard deviation σ_l = 0.2 mm/year is shown in Fig. 3.16. As mentioned above, it is not possible to use different apriori standard errors in the present version of Kriging. The resulting solution is called the *smoothing Kriging* (SURFER 8) model below.

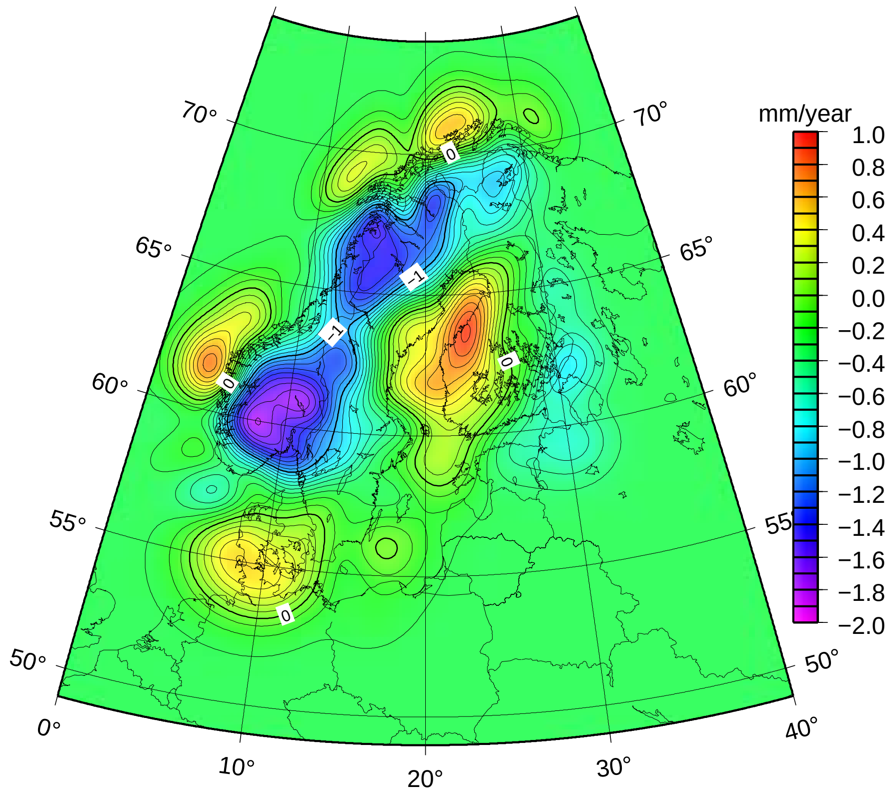

Figure 3.16: Contour lines for the difference between the *smoothing Kriging* and *Lambeck* models. Unit: mm/year.

It can be seen that the result looks much better. No clear oscillations can be discerned and the figure reminds of the smoothing inverse distance counterpart in Fig. 3.11. However, notice the "hills" outside the Norwegian coast, but more of this later (see below). It should further be noticed that the model do not exactly approach the mean value shifted Lambeck far away from the observations. It gives -2.32 mm/year instead of -2.68 mm/year. It seems like numerical effects are responsible. One could otherwise expect that since no spatial correlation occurs far from the observations, the predicted value should be exactly equal to the mean; cf. the discussion above. However, this problem is of little practical significance. As will be seen in Section 3.3, we nevertheless have to specify the minimum value of the resulting model to a larger value than both -2.68 mm/year and -2.32 mm/year to obtain a realistic value in Amsterdam (NAP), which is important in the present project. As the solution behaves well close to the observations, the difference far away is therefore considered unproblematic.

The least squares collocation solution is made using the GEOGRID software (Forsberg 2003). Vestøl's estimated standard errors are applied to construct the diagonal of the matrix $\boldsymbol{D}$ and it is assumed that the observations are uncorrelated. To speed up the computations, the GEOGRID search algorithm (see Forsberg 2003) is taken advantage of using 25 observations per quadrant. It is of course also possible to tune the FORTRAN parameter statements so that all observations are utilised, but due to the severe time limitations concerning the finalisation of RH 2000 the default settings are preferred here. GEOGRID utilises a second order Markov process covariance function, defined as

$$C(d) = C_0 \left(1 + \frac{|d|}{\alpha} \right) \exp\left\{ -\frac{|d|}{\alpha} \right\} \tag{3.11}$$

where α is a parameter related to the correlation length as $\alpha \approx 0.595 \cdot d_{1/2}$. The variance C_0 is automatically estimated from the data. The correlation length is chosen to 165 km. This is practically equal to the value implied by Eq. (3.9) and (3.10). Notice also that the remove-compute-restore estimator in Eq. (3.3) is applied, which implies that the mean value is first removed and later restored. The GEOGRID solution will be referred to as the *smoothing collocation* (GEOGRID) model below. The difference from Lambeck (without mean value shift) is illustrated in Fig. 3.17.

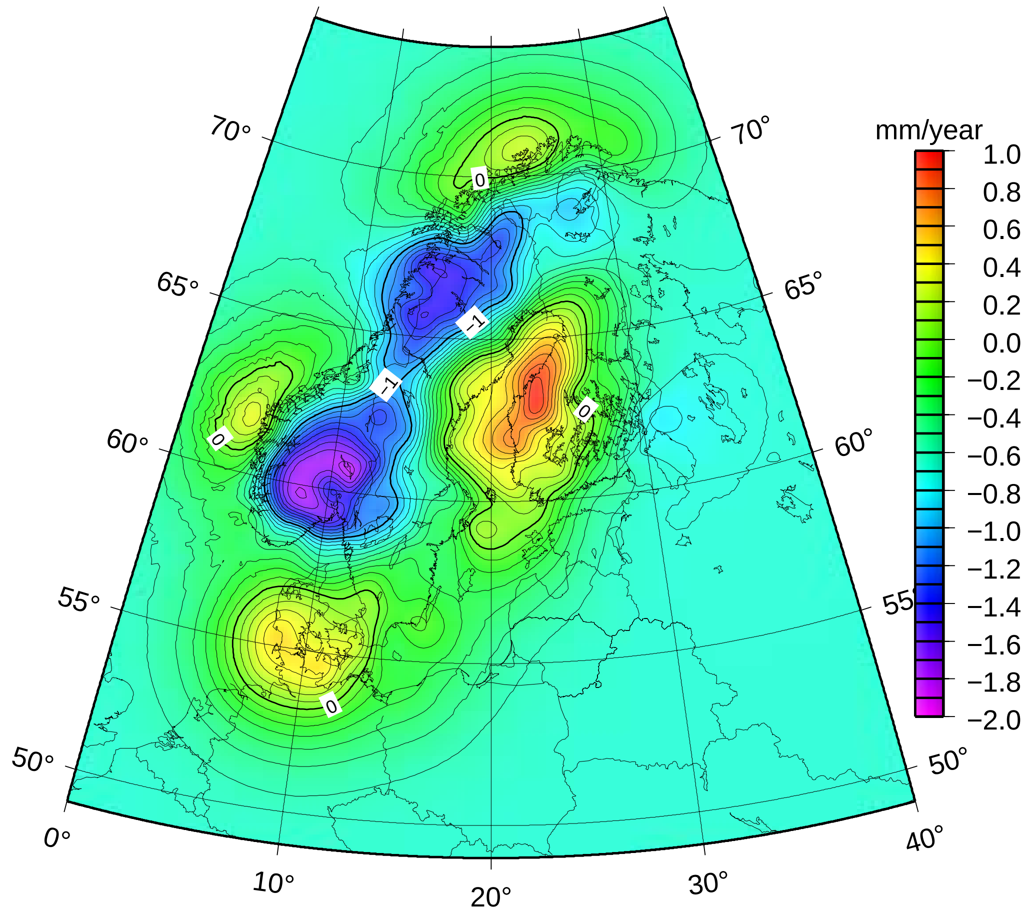

Figure 3.17: Contour lines for the difference between the *smoothing collocation* and *Lambeck* models. Unit: mm/year.

If Figs. 3.16 and 3.17 are compared, it is clear that the smoothing Kriging and smoothing collocation solutions are very similar, especially in areas with observations. The most notable difference can be discerned outside the coast of Norway, where the oscillations are considerably smaller in the collocation case compared to the Kriging solution above. This depends on the outermost uplift observations in question being often of questionable quality, due to the fact that open levelling lines are used and/or only limited land uplift information being available; cf. the discussion at the end of Subsection 2.2.1. These deficiencies are naturally reflected in the estimated standard errors; see Fig. 2.12. As the "hills" outside the Norwegian coast in Fig. 3.16 all occur outside point observations with high standard errors, it is not strange that the least squares collocation solution yields smaller oscillations. Another difference between the two models is that the collocation solution approaches mean value shifted Lambeck far from the observations, exactly as predicted above, while this is only approximately fulfilled for the

Kriging solution. It can also be seen in Fig. 3.17 that the GEOGRID search algorithm results in some zigzag behaviour of the contour lines, but this effect is so small that it can be neglected in the present context. The search algorithm works well and 25 observations per quadrant are obviously sufficient for the purpose.

However, neglecting the small differences discussed above, the main conclusion is that the two models are similar. Due to this fact, it seems sufficient to analyse only one of them in detail. Rather arbitrarily, the smoothing Kriging model is therefore chosen. The model is illustrated in Figs. 3.18 and 3.19.

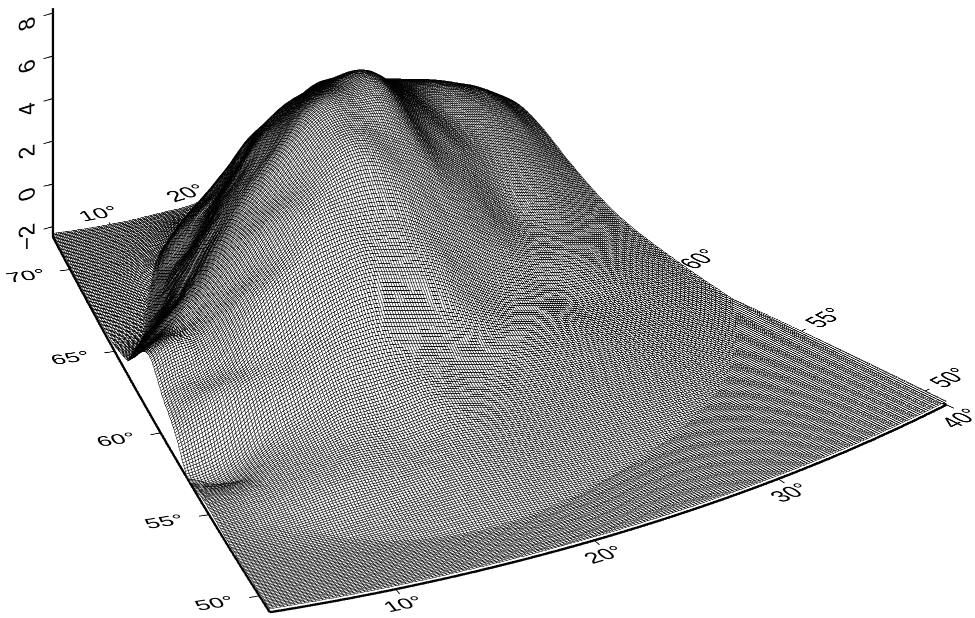

Figure 3.18: Vestøl's point model above the dividing line extended with Lambeck. Smoothing Kriging interpolation/extrapolation. Unit: mm/year.

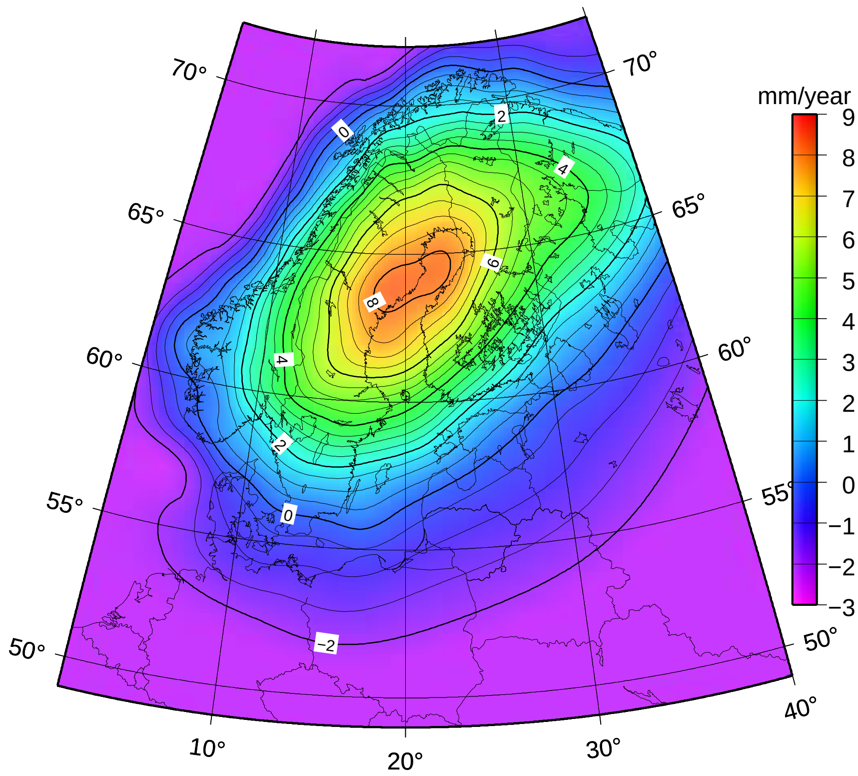

Figure 3.19: Contour lines for Vestøl's point model extended with Lambeck. Smoothing Kriging interpolation/extrapolation. Unit: mm/year.

It can be seen in the above figures that the smoothing Kriging model is smooth and “looks” as realistic as the smoothing inverse distance counterpart above. Before the difference between inverse distance interpolation and Kriging/collocation is considered, let us study how well the Kriging model fits with the observations and Vestøl’s point values. The corresponding statistics is presented in Table 3.4.

The smoothing Kriging model fits equally well to the GPS and tide gauge observations as the smoothing inverse distance counterpart, but is considerably closer in the observation points. The RMS value reduces from 0.20 mm/year in the inverse distance case (Table 3.2) to 0.11 mm/year for Kriging. Considering the smooth character of the solution and that it reproduces Vestøl’s point values so well, makes Kriging (or least squares collocation) a strong candidate for the final solution.

Table 3.4: Statistics for the apparent uplift residuals for Vestøl's point model interpolated using the smoothing Kriging method. The maximum for "All tide gauges" is given for both the outlier stations Furuögrund/Oslo. Unit: mm/year.

Observations	#	Min	Max	Mean	StdDev	RMS
Vestøl's point values above the dividing line	743	-0.51	0.43	0.00	0.11	0.11
All tide gauges	58	-0.60	0.78/1.33	0.03	0.27	0.27
Edited tide gauges	56	-0.60	0.38	-0.01	0.18	0.18
All GPS	55	-1.31	1.79	0.09	0.59	0.59
SWEPOS GPS	21	-0.64	0.44	-0.05	0.30	0.30

However, it is not certain that we want to have the best possible fit to all observations. One unfortunate property of the present Kriging solution is that it might not have been filtered sufficiently at the borders of "observation areas". In such cases, no information is available on one side (so to speak) and the estimate becomes more dependent on the outermost observation. In such cases, the interpolated surface is also likely to deviate systematically outside the border in question; cf. the hills outside the Norwegian coast in Fig. 3.16. Of course, the phenomenon is reduced when least squares collocation with individual apriori standard errors is utilised, but it occurs also in this case. In this respect, the present inverse distance method provides a more effective filtering. Let us consider a portion of the Norwegian coast as an example, in which some land uplift observations of bad quality are present; cf. Figure 2.12. The deviation between the point values (observations) and the smoothing inverse distance as well as the smoothing Kriging observations are presented in Fig. 3.20. It can be seen in Fig. 3.12 that the differences from Lambeck deviate in some of the outermost points compared to the general trend in the area. This depends on the land uplift being not well determined in these points. Actually, the uplifts in the point close to the left corner and in the two most northern points have been extrapolated using a fifth degree polynomial; cf. the discussion at the end of Subsection 2.2.1. What happens with the Kriging solution in this case is that the model fits too well, while the smoothing inverse distance interpolation filters the observations more appropriately.

The same type of example can be constructed also in Sweden, for instance along the coasts of the Baltic Sea. It is thus concluded that the filtering properties along the "observation borders" are better for the present smoothing inverse distance solution compared to the Kriging counterpart.

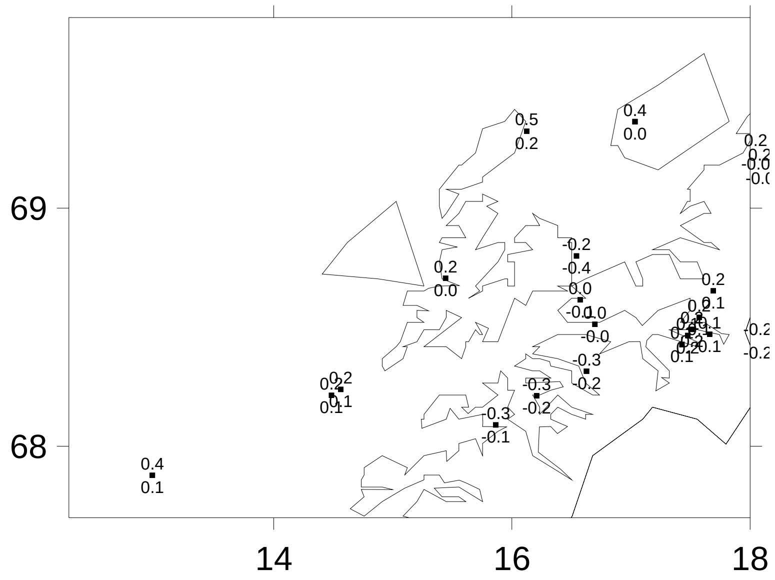

Figure 3.20 Difference of the Vestøl's point observations from the smoothing inverse distance (upper value) and smoothing Kriging (lower value) models. Unit: mm/year.

3.2.4 Choice of interpolation method

At the end of the last section it was concluded that the filtering properties of inverse distance interpolation (with the chosen parameters) are arguably better close to borders, but as discussed in Subsection 3.2.2, this type of interpolation also has its disadvantages. The question now is which method that should be chosen for the final model. In the same way as in the last subsection, it will be approached by first studying how much the corresponding models differ. It is then investigated how dependent the adjusted heights of the Baltic Levelling Ring are on the choice.

The differences between the smoothing Kriging and smoothing inverse distance models are illustrated in Fig. 3.21. The most notable thing in Fig. 3.21 is that it is of a comparatively long-wavelength nature. The effect discussed in the last subsection, that the inverse distance model is too high in the "convex upward" areas with a positive second derivative (southern Norway) and vice versa (western Finland), can be clearly discerned. Otherwise, the largest differences between the models occur outside the coast of Norway and in the Baltic. It is interesting to note that several of the "hills" outside the coast of Norway occur outside low quality observations; cf. Fig. 2.12. Even though these effects are reduced by the least squares collocation solution, which uses the estimated standard errors for the weighting, they are significant also in this case. It is therefore believed that the smoothing inverse distance interpolation is a little more robust in the areas without observations.

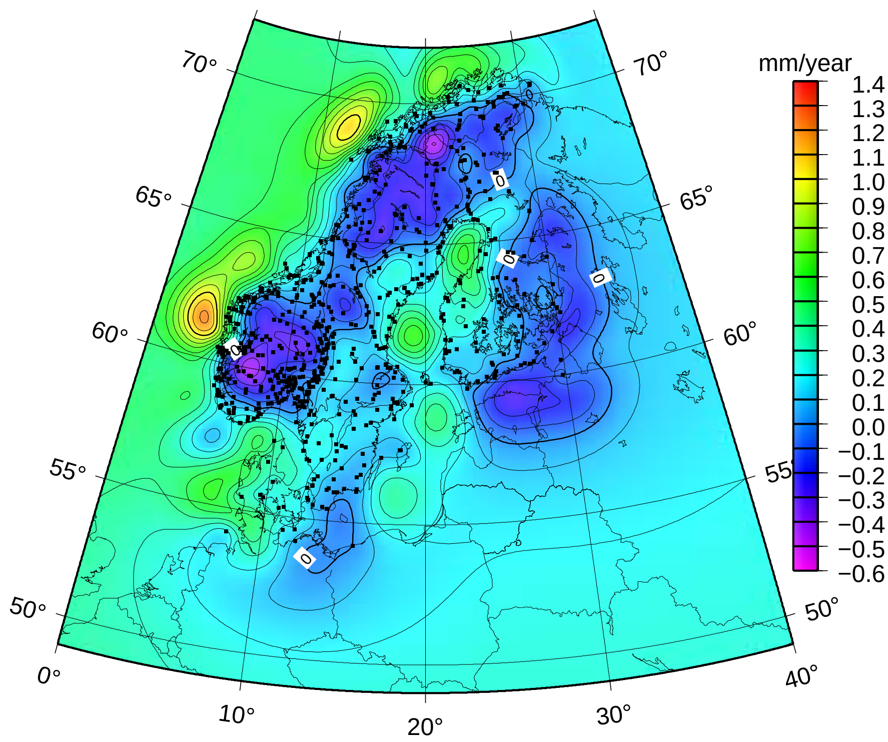

Figure 3.21: Contour lines for the difference between the *smoothing Kriging* and *smoothing inverse distance* interpolation models. The dots denote Vestøl's point values above the dividing line. Unit: mm/year.

Another thing that can be seen in Fig. 3.21 is that the two models agree well in Sweden. The differences are typically 0.1 – 0.2 mm/year. In addition, they are of a long-wavelength nature. Considering the standard errors of the uplift in Sweden in Fig. 2.12, it is clear that differences between the methods are significantly smaller than the standard errors of the observations. Let us now turn to how dependent the adjusted heights are on the differences in question. The difference between using the smoothing Kriging and smoothing inverse distance models in the adjustment of the Baltic Levelling Ring is illustrated in Figs. 3.22 and 3.23 for the whole area and Sweden, respectively. The corresponding statistics can be found in Table 3.5.

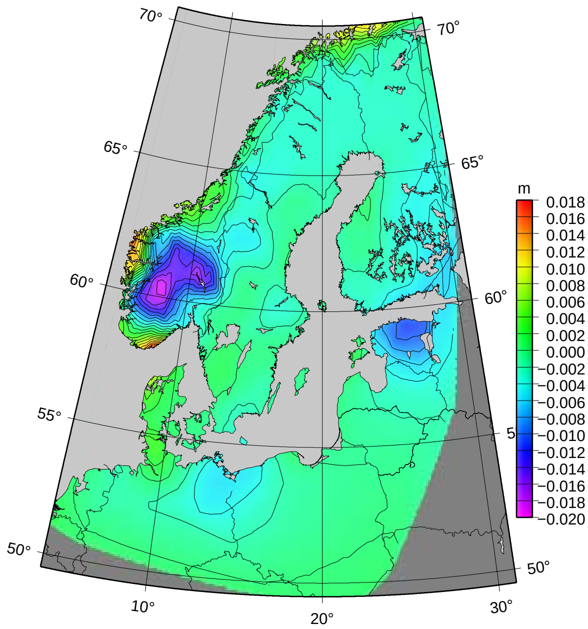

Figure 3.22: Adjusted height differences between using the *smoothing Kriging* and *smoothing inverse distance* interpolation models for the Baltic Levelling Ring. Delaunay triangulation with linear interpolation used for the visualisation. Unit: m.

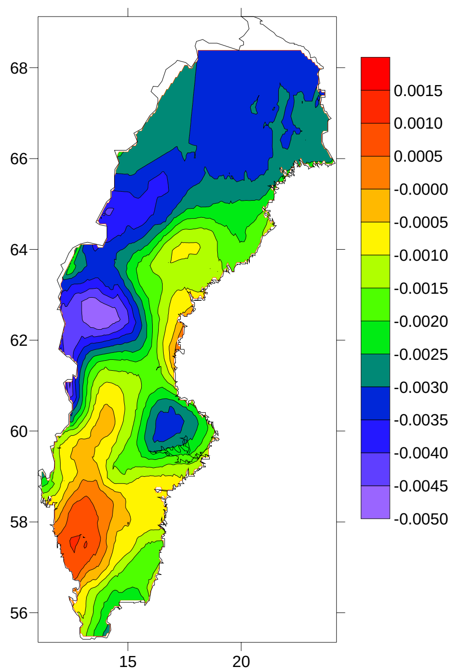

Figure 3.23: Adjusted height differences between using the *smoothing Kriging* and *smoothing inverse distance* interpolation models for RH 2000 in Sweden. Delaunay triangulation with linear interpolation used for the visualisation. Unit: m.

Table 3.5: Statistics for the difference in adjusted heights between using the *smoothing Kriging* and *smoothing inverse distance* uplift models. Unit: m.

Observations	#	Min	Max	Mean	StdDev	RMS
BLR	7401	-0.0196	0.0182	-0.0016	0.0021	0.0026
RH 2000	5088	-0.0050	0.0011	-0.0016	0.0012	0.0020

It should be noticed that the difference in mean value depends on the two interpolation methods yielding somewhat different results for the NAP in Amsterdam, but as this value has to be modified anyhow, this should not bother us here; see the next section. The most interesting thing in Fig. 3.22 is that the choice of model is most important in Norway, exactly as in Subsection 3.2.2, which again depends on the many old levelling lines. Notice the systematic difference in the mountains in the southern parts of the country, which depends on the bias of the smoothing inverse distance method. Another observation is that the levelled heights differ in the most northern parts, which depends on the different filtering properties of the two interpolators. As can be seen in Fig. 3.20, a number of the outermost observations are filtered differently. As some of the involved levelling lines are old, differences up to two centimetres are generated in the most northern levelling benchmarks.

In Sweden, the difference between the methods is only a few millimetres and of long-wavelength nature, which is reassuring. The maximum height difference (highest minus lowest) is 6.1 mm within the whole country and the standard deviation is 1.2 mm. By comparing the above results with Subsections 3.2.1 and 3.2.2, in which the exact and smoothing inverse distance methods were studied, it may be concluded that it is more crucial whether an exact (Vestøl's original) or a smoothing interpolation is used. It is consequently not too important which smoothing alternative that is chosen. As we nevertheless have to choose one interpolation/extrapolation technique, the *smoothing inverse distance method is preferred*. The main reason for this choice is that it is considered as having somewhat better filtering properties at the "observation borders". As Vestøl's model relies on the land uplift from the second and third precise levelling for large regions (see Section 2.1), it is believed that it is important to use a method that filters the point values effectively, also at the “borders” of the observations (for instance in the central parts of northern Sweden). Of course, we have to pay the price that the land uplift model is slightly biased, but this effect is hardly significant in Sweden, where it is only a few millimetres. It is more important in southern Norway, but in this area of non-repeated levelling the quality of the uplift is questionable anyhow. It might even be argued that the bias in question is positive in such areas, since it might damp the magnitude of large systematic errors somewhat; cf. the discussion in the last section. Another less important argument for using the inverse distance method is that it seems a little more robust in areas without observations, for instance in the Baltic and Norwegian Seas. It is true that this is not important for the adjustment of the Baltic Levelling Ring, but the resulting uplift model might be used also in other areas in the future.

It is admitted that the Kriging (collocation) solution is more suitable in other respects: It has for instance the advantage (or disadvantage) that it fits better with all of Vestøl’s point observations at the same time as it is as smooth as the inverse distance counterpart. However, it should be pointed out that a large number of other interpolation methods exist and that it is more or less impossible to judge which one that is most optimal in the particular case. Furthermore, both the inverse distance and Kriging (collocation) methods have been tuned in different ways, which is also arbitrary to some extent. Thus, it will always be uncertain whether we have chosen exactly the correct method and parameters. In the present case, it is the authors’ belief that we have reached a point where the two smoothing strategies both can be considered as sufficiently good. Based on the reasons mentioned in the last paragraph, the smoothing inverse distance method is therefore chosen as interpolation and extrapolation method for the adjustment of the third Swedish precise levelling.

3.3 Closing errors around the Gulf of Bothnia and the Baltic Sea

Let us finish this chapter by presenting two inconclusive evaluations of the above interpolation methods, which were made by studying the closing errors around the Gulf of Bothnia and the whole Baltic Sea. The main purpose is to show how dependent the closing errors are on the choice of interpolation method and to investigate whether everything is in order with the adjustment. Due to the propagation of random levelling errors, and due to possible errors in the connections across the Gulf of Finland and the Åland Sea, it is difficult to say with certainty that one interpolation method is better than the other. It is consequently not the aim to use the results as an arbiter in the choice of interpolation method.

It should first be emphasised that only levelling observations are used in all the adjustments of the Baltic Levelling Ring that are presented in this report. This means that no connection is utilised between Sweden and Finland across the Åland Sea and that Finland is not directly tied to the network in Estonia. However, by taking advantage of other information than levelling, it becomes possible to compute the closing errors. The closing error around the Gulf of Bothnia can be computed using the oceanographic estimate in Ekman and Mäkinen (1996b) of the mean sea level difference at tide gauges on opposite sides of the Åland Sea. In a similar way, the closing error around the whole Baltic Sea can be found using the connection across the Gulf of Finland established by Jürgenssen and Saaranen (personal communication). The latter is based on a combination of GPS, the NKG 2002 geoid model (Forsberg and Strykowsky, personal communication) and levelling to several tide gauges. Of course, the accuracy of these connections is somewhat questionable, which is one reason for not including this information in the Baltic Levelling Ring adjustments. Now, the closing errors for the exact inverse distance, smoothing inverse distance and smoothing Kriging uplift models are presented in Table 3.6.

Table 3.6: Closing errors for different land uplift models using alternative connections across the Åland Sea and the Gulf of Finland. Unit: m.

Land uplift model	Åland Sea	Gulf of Finland
Exact inverse distance (original Vestøl)	0.0038	0.0148
Smoothing inverse distance	0.0183	0.0110
Smoothing Kriging	0.0202	0.0070

If it is considered that the levelling extends over thousands of kilometres, and that the levelling lines between Finland and Åland include optical water crossings, the closing errors in Table 3.6 are incredibly small. This shows that everything is in order with the adjustment. However, as stated above, it is not possible to judge which interpolation method that performs best. It might for instance be noted that for the connection across the Åland Sea, the exact inverse distance differ considerably from the other two methods, which is in agreement with the previous conclusions. However, this result is contradicted by the closing errors across the Gulf of Finland. Considering the many uncertainties, nothing should thus be concluded in this respect. To sum up, the most important conclusions that can be reached from Table 3.6 are that the differences between the interpolation techniques are small (1 cm level) for the very long distances in question and that everything seems to be fine with the adjustment of the Baltic Levelling Ring.

4. The choice of final uplift model and its consequences

As discussed in the introduction in Chapter 1, the system definition for RH 2000 includes five major components, namely the land uplift model, reference epoch, zero level, type of heights and system for the permanent tide. Since RH 2000 is defined to be a Swedish realisation of EVRS (European Vertical Reference System) as defined in 2005, only the first two remained to be chosen on a national level. The reference epoch for the land uplift corrections should preferably be chosen to the mean of all observations, but due to more political aspects of the problem, it was nevertheless specified to 2000.0. Consequently, the remaining and most important part is the construction of a suitable land uplift model. However, this does not mean that all other components of the definition can be totally neglected. The construction of a land uplift model is related to the specification of zero level through the fact that the Normaal Amsterdams Peil (NAP) is affected by the uplift; it sinks with respect to the geoid. One important question at this point is whether this effect should affect the zero level or not. To answer this question, it is important to carefully outline and discuss the 2005 definition of EVRS. This is one purpose of Section 4.1.

In the last chapter it was investigated how Vestöl's and Lambeck's land uplift models should best be combined. It was mentioned several times that the situation is not completely satisfactory at the outskirts of the model, for instance in the vicinity of the NAP. Another purpose of Section 4.1 is to modify the model so that it becomes more realistic in its non-central parts. After that, the final land uplift model is presented and analysed (Section 4.2). It is compared how well the model fits to the available observations and how much the final RH 2000 heights differ from the heights that would have been obtained in case Lambeck's model would have been applied without modification.

The last two sections deals with the consequences of the system definition, which is regarded as including the land uplift model. In Section 4.3 the main topic is to investigate how the resulting RH 2000 heights relate to the old Swedish height system RH 70, to the European Vertical Reference Frame EVRF 2000 and to the new

Danish height system DVR 90 (Schmidt 2000). The other Nordic countries (Finland and Norway) have not yet finalised their new height systems/frames. Section 4.4 finally deals with the question where the Mean Sea Level (MSL) is located in RH 2000 at four mareographs along the Swedish coast.

4.1 Definition of RH 2000 and the land uplift in Amsterdam (NAP)

In the last chapter it was decided to use the smoothing inverse distance method to interpolate and extrapolate Vestøl's point values. As discussed in Section 3.1, the inverse distance method approaches the mean value far away from the observations. If the remove-compute-restore technique is applied with respect to Lambeck's model, it follows that the extrapolated model approaches the mean value shifted Lambeck (-2.68 mm/year) at large distances from the uplift area. One problem here is that the apparent uplift for the Normaal Amsterdams Peil (NAP) arguably becomes too low, namely -2.54 mm/year. Using Vestøl's estimated value of the eustatic sea level rise (-1.32 mm/year), the corresponding land sinking is 1.22 mm/year with respect to the geoid. Admittedly, this figure is in the right neighbourhood, but considering both the literature and the available observations, the sinking is a bit too large. For instance, the apparent uplift in the NAP is -2.0 mm/year according to the model of Milne et al. (2001); see also Mäkinen (2004). Moreover, the two GPS stations in the vicinity of Amsterdam give the mean value -1.9 mm/year apparent uplift (see Fig. 2.9).

In Section 2.3 it was mentioned that Lambeck's (digitised) uplift model was extended in such a way that the minimum apparent uplift becomes exactly -2.00 mm/year; see Fig. 2.14. Of course this method implies that the resulting model cannot be perfect in its outermost parts. It is well known that the uplift (in relation to the geoid) first becomes slightly negative and then smoothly approaches zero far away from the uplift centre. Considering the estimated eustatic sea level rise, it follows that Lambeck's model should approach -1.32 mm/year after a certain (unknown) distance. However, if it is assumed that the model is used no further than NAP and that the uplift is realistic thus far, then the model is nevertheless good for the present purpose in case the minimum value is representative. One drawback is the discontinuity of the first derivative when the model reaches the bottom (so to speak), but considering the many other uncertainties involved, this effect seems like a minor problem. Now, the strategy used to extend Lambeck's model will be adopted also for the RH 2000 model and the main question is how the minimum value should be chosen. This might seem like a straightforward enough matter, but as it is intimately related to the system (datum) definition of RH 2000, it is important to investigate how the specification of the NAP uplift affects the final heights for RH 2000. These questions are investigated in this section.

The 2005 definition of the European Vertical Reference System (EVRS) and the definition of RH 2000 were briefly discussed in the introduction (Chapter 1). Let us now consider this topic in more detail. Concerning the European systems we try to follow the standard IERS terminology that differentiates between reference systems and frames: the former is the definition, while the latter refers to the realisation. However, as no such distinction exists in Sweden, we prefer to use the Swedish convention and talk about the Swedish "reference system" RH 2000. The term "reference system" refers to both the definition *and* its realisation. What is meant should be clear from the context. Now, RH 2000 is *defined* to be the Swedish realisation of the European Vertical Reference System (EVRS) in 2005, which is defined in the following way (quoted from the EUREF home page in 2005; see also Ihde and Augath 2001 and Mäkinen 2004):

- "The vertical datum is the zero level for which the Earth gravity field potential W_0 is equal to the normal potential of the Mean Earth Ellipsoid U_0:

$$W_0 = U_0 . \tag{3.12}$$

- The height components are the differences ΔW_P between the potential W_P of the Earth gravity field through the considered points P and the potential of the EVRS zero level W_0. The potential difference $-\Delta W_P$ is also designated as geopotential number c_P:

$$-\Delta W_P = W_0 - W_P = c_P \tag{3.13}$$

Normal heights are equivalent to geopotential numbers.

- The EVRS is a zero tidal system, in agreement with the IAG resolutions."

Notice the quotation marks. Admittedly, this sounds like a definition of a World Height System (WHS). What is specifically European, however, is the way EVRS is realised. The latest realisation of EVRS (in 2005), which is called the European Vertical Reference Frame EVRF 2000, is characterised by the geopotential numbers and normal heights for the nodal benchmarks of the United European Levelling Network 95/98 (UELN 95/98) in relation to the Normaal Amsterdams Peil (NAP). The realisation is made according to the following conventions (again the point list is inside quotation marks):

- "The vertical datum of the EVRS is realized by the zero level through the Normaal Amsterdams Peil (NAP). Following this, the geopotential number in the NAP is zero:
$$C_{NAP} = 0 .$$
- For related parameters and constants of the Geodetic Reference System 1980 (GRS 80) is used. Following this, the Earth gravity field potential through NAP W_{NAP} is set to be the normal potential of the GRS 80
$$W_{NAP}^{realisation} = U_0^{GRS80} .$$
- The EVRF 2000 datum is fixed by the geopotential number and the equivalent normal height of the reference point of the UELN No. 000A2530/13600."

Thus, the EVRS is realised through the zero level in NAP, which refers to the sea level (mean high tide) in 1684. This is obviously problematic in many respects; see Mäkinen (2004) for a longer discussion. It seems best to view the height of the NAP reference point as a convention, which is valid independently of time. It is irrelevant where the sea surface was back in 1684. It should further be noticed that no motion is specified for NAP, even though it is clear that the sea level rises with approximately 2 mm/year in Amsterdam (more than the eustatic sea level rise); see Mäkinen (2004) and the above discussion of apparent uplift at the NAP. There is also evidence that the reference point is locally unstable (Mäkinen ibid.). This means that different realisations will not refer to the same equipotential surface. Instead they are determined relative to the physical reference point 000A2530/13600. Thus, since the NAP is (very likely) moving, different realisations "realise" EVRS differently. This might be the reason for including the specification of the NAP zero level in the realisation (or frame) part of the description above, which is otherwise somewhat strange, since it seems like a crucial part of the system definition. Of course, the EVRS definitions in 2005 might be interpreted as saying that any system satisfying the very general system requirements above constitutes a realisation, but since this implies that almost any gravity related vertical system/frame qualifies as a realisation, we view EVRS as implicitly defined using the NAP zero level. We prefer to finish this rather philosophical discussion at this point. The question how the EVRS/EVRF should best be defined in the future is discussed at length in Mäkinen (2004). For the time being (2005) we simply have no choice but to define RH 2000 in analogue to EVRF 2000 using the NAP reference. This is in accordance with how other "NAP" countries like Germany and the Netherlands have realised their height systems.

The EVRS is specified to use a zero system for the permanent tide (e.g. Ekman 1989). It should be noticed, though, that the conventional zero level in NAP was used independently of permanent tide systems for many years. Only recently, after the question was brought to focus by Ekman (1989) and others, the EVRS was specified to be a zero system. To our knowledge, no correction was applied to convert the original NAP level to such a system. This means that the same NAP value has been used to fix height systems that treat the permanent tide differently. For instance, the Swedish height system RH 70 utilises a non-tidal permanent tide system, but has been realised using the same NAP value as cited above.

To sum up, the EVRS (in 2005) is realised by keeping the NAP (the reference benchmark 000A2530/13600) fixed, but since this reference is moving, realisations made at different times deviate from each other, also in case all relative motions are modelled properly. This means that if one frame is to be converted to another with different epoch, it is sufficient to reduce the levelling observations (relative height differences) to the new epoch. No correction should be applied to the NAP height. Furthermore, the EVRS of today is a zero permanent tide system, but previous versions were unspecified in this respect. When the same NAP value has been used to fix two height reference systems/frames with different treatments of the permanent tide, no correction should be applied to the NAP height when transforming between the corresponding tide systems. It is sufficient to transform the height differences relative to the NAP.

As stated in the introduction, the new Swedish reference system RH 2000 is defined to be a realisation of EVRS. This implies that a zero permanent tide system and normal heights are to be used. According to the above discussion, it also implies (for the time being, i.e. 2005) that the zero level is fixed by means of the NAP. Even though this reference is not stable, the NAP should nevertheless be treated as if it did not move at all. The potential number of the reference point No. 000A2530/13600 cited above is consequently fixed in the adjustment of the Baltic Levelling Ring. The internal vertical movement caused by the Fennoscandian land uplift, on the other hand, is reduced to the reference epoch 2000.0 using the model of this report. This epoch is not given as part of the EVRS definition, but was chosen for its "political correctness"; see Chapter 1.

Since the land uplift model is only used to correct levelling lines, which involve the uplift at two different stations, only the uplift difference with respect to the NAP is really used. As a consequence, the adjusted heights in Sweden move up and down depending on how the NAP uplift is chosen in our model, but it is not known how dependent the adjusted heights are on this choice. As mentioned above, the smoothing inverse distance model estimated from Vestøl's point values (illustrated in Figs. 3.9 to 3.11) is to be modified so that a suitable uplift value is obtained in NAP. The method is the same as was applied for Lambeck's model in Section 2.3, namely to redefine the minimum value to a suitable value. To see how sensitive the estimated heights are, the height differences between using the model illustrated in Fig. 4.1 with the minimum value -2.00 mm/year and the original smoothing inverse distance model with the minimum value -2.68 mm/year are presented in Fig. 4.2. The corresponding apparent uplift values in the NAP are -2.54 and -2.00 mm/year, respectively.

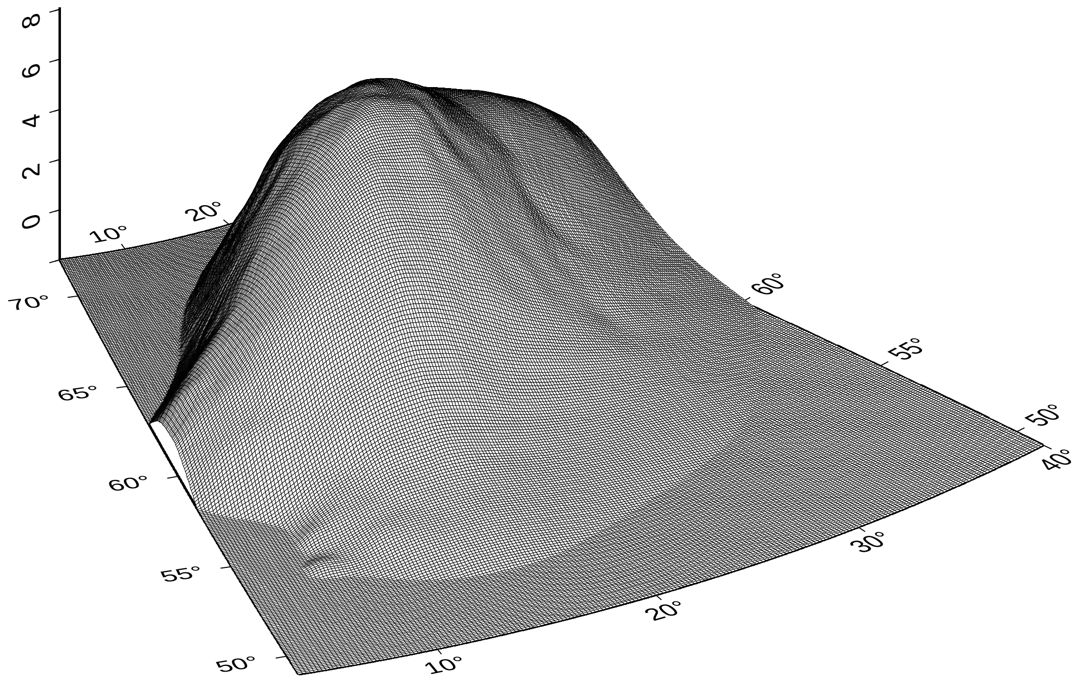

Figure 4.1: Apparent uplift for the smoothed inverse distance model with minimum value -2.00 mm/year. Unit: mm/year.

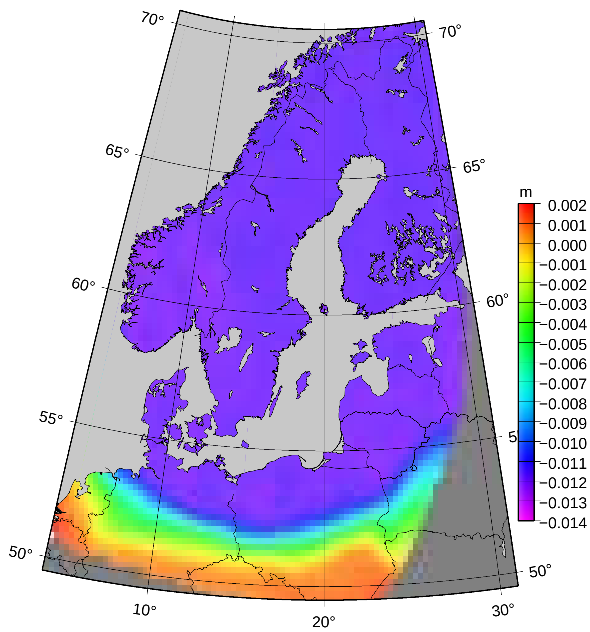

Figure 4.2: Adjusted height differences between using a smoothed inverse distance model with minimum values -2.68 and -2.00 mm/year. Unit: m.

It can be seen from Fig. 4.2 that the adjusted heights in the Nordic countries reduce by approximately 13 mm when the apparent uplift in the NAP changes from -2.54 mm/year to -2.00 mm/year. The effect is systematic (on the 1 mm level) and may be viewed as inducing a systematic shift over Sweden. It is thus concluded that the resulting heights are not too dependent on the NAP uplift. Since it will be impossible to specify the "true" uplift in the NAP, we can expect an error in the form of a systematic shift on the 1 cm level. As we nevertheless have to choose something, the minimum value is taken as -2.00 mm/year, which agrees with the model in Milne et al. (2001) and the GPS observations in the Netherlands; see Fig. 2.9. Considering the eustatic sea level rise, this means that the land sinking in Amsterdam is 0.68 mm/year with respect to the geoid, which is reasonable. The smoothing inverse distance model with the minimum -2.00 mm/year is taken as the *final land uplift model for RH 2000*. It is further investigated in the next section.

4.2 The RH 2000 land uplift model: NKG2005LU

In what follows the land uplift model used for the computation of RH 2000 will be referred to as NKG2005LU. (Earlier this model was called RH 2000 LU; see the note in Sect. 1.5). In this section NKG2005LU is first presented. It is then investigated how well it fits with the observations. It is finally tested how much the adjusted heights of the Baltic Levelling Ring differ depending on whether the model of Lambeck et al. (1998) or NKG2005LU is applied to model the land uplift.

The contour lines for the final model NKG2005LU can be found in Fig. 4.3. The residuals for the GPS and tide gauge observations are presented in Fig. 4.4 and the usual statistics is given in Table 4.1. A black and white version of Fig. 4.3 can be found in Chapter 5.

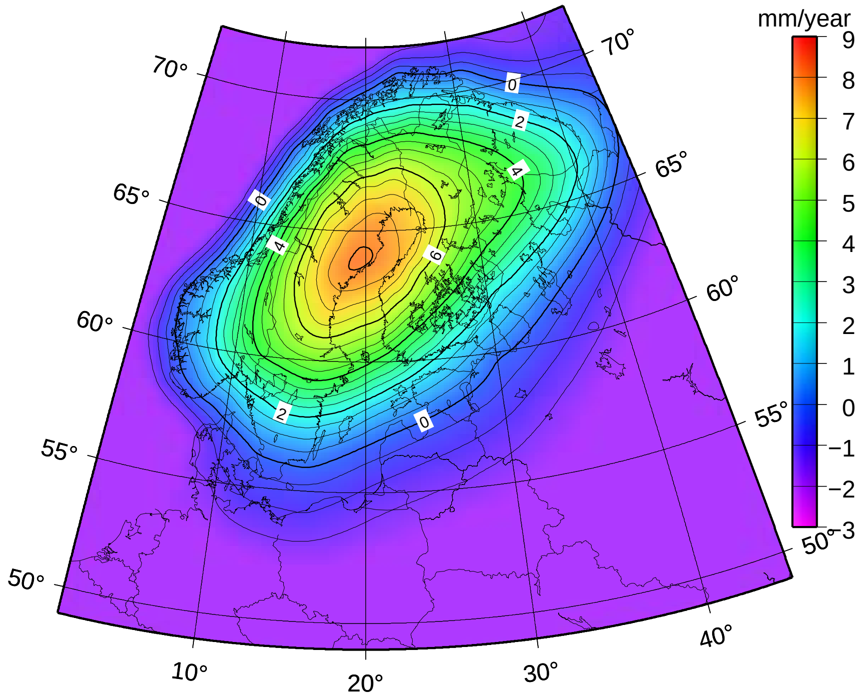

Figure 4.3: Contour lines for the apparent uplift from the RH 2000 land uplift model NKG2005LU (Smoothed inverse distance model with minimum value -2.00 mm/year). Unit: mm/year.

Table 4.1: Statistics for the residuals of NKG2005LU (smoothed inverse distance model with minimum -2.00 mm/year). The maximum for "All tide gauges" is given for both the outlier stations Furuögrund/Oslo. Unit: mm/year.

Observations	#	Min	Max	Mean	StdDev	RMS
All tide gauges	58	-0.36	0.93/1.24	0.18	0.26	0.31
Edited tide gauges	56	-0.36	0.55	0.14	0.19	0.23
All GPS	55	-1.15	1.46	0.13	0.52	0.53
SWEPOS GPS	21	-0.59	0.49	0.03	0.32	0.32

It can be seen that NKG2005LU fits well with the observations. For both the mareograph and GPS observations, the RMS values are in agreement with the standard errors (see Section 2.1), at the same time as the model is smooth; cf. Fig. 4.1. Another thing that can be noticed is that the residuals of the SWEPOS stations in the central parts of Sweden are much lower compared to the same errors for Lambeck's model; cf. Section 2.3. Otherwise, most features of the final model have already been discussed, for instance the exclusion of the mareographs in Furuögrund and Oslo; see Subsection 2.2.2. The most important aspect to consider here is the behaviour of NKG2005LU in the Netherlands, Germany and Poland, which is affected by the choice of minimum value (-2.0 mm/year), which was discussed in the last section. As can be seen, the model agrees reasonably with the GPS observations as far out as to the Netherlands and middle Germany and middle Poland. South of that, it is clear that NKG2005LU gives too small uplift values, exactly as expected considering the way the model was constructed. The four southernmost residuals are 1.5, 0.7, 1.3 and 1.4 mm/year; see Fig. 4.4. However, since NKG2005LU is not meant to be applied in this area, this is not considered problematic in the present context; cf. the discussion in Section 3.3.

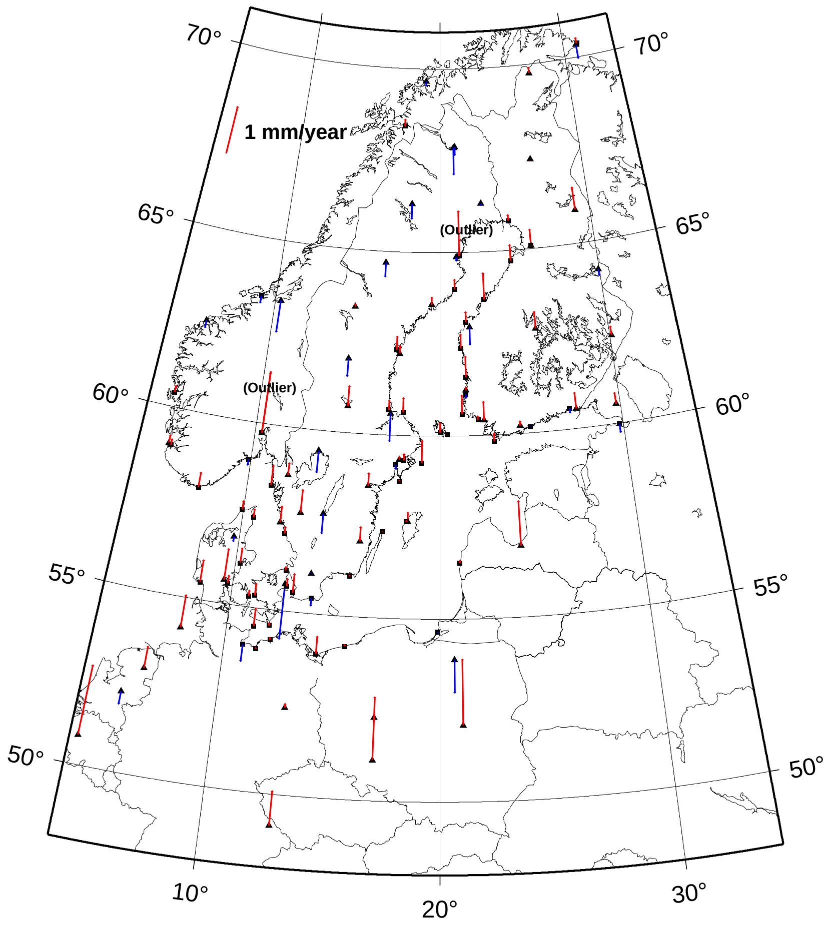

Figure 4.4: Mareograph and GPS residuals for NKG2005LU (smoothed inverse distance model with minimum value -2.00 mm/year). Unit: mm/year.

It was mentioned in the introduction that it was at one time seriously considered within the NKG to utilise Lambeck's model in unmodified shape. From the Swedish point of view, the main problem with this would have been the poor fit to the GPS observations in the central parts of the country; see Fig. 2.16. Up to now, we have followed a long road to reach the final uplift model, which is believed to be a good combination of Vestöl's and Lambeck's models. An important question at this point is how much the adjusted heights are affected by the difference between NKG2005LU and the model of Lambeck, i.e. how much we have gained by our efforts to improve Lambeck's model. The land uplift difference itself was illustrated in Fig. 3.11 for the central parts of the area. Due to the modification of the minimum value of the smoothing inverse distance model (leading to NKG2005LU), the difference at the outskirts of the area now vanish. The adjusted height differences between using the two models are presented in Fig. 4.5.

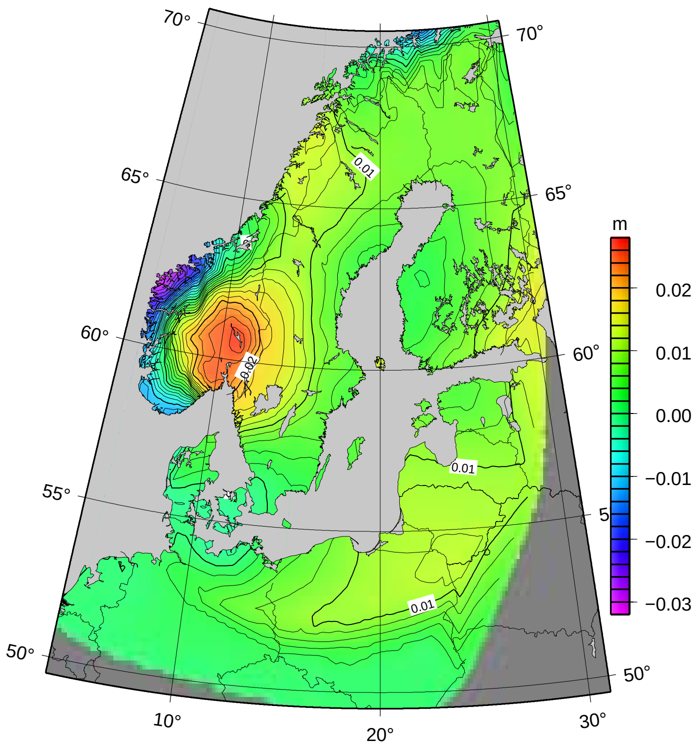

Figure 4.5: Adjusted height differences between using the land uplift model of Lambeck et al. (1998) and the NKG2005LU model. Delaunay triangulation used for the visualisation. Unit: m.

By studying Fig. 4.5, it can be seen that the differences are on the cm level and of a comparatively long wavelength nature. As in all similar comparisons in Chapter 3, the largest discrepancies occur in Norway. In Sweden, they reach approximately 2 cm at the Norwegian border. The main effect in Sweden is otherwise a slope in the east-west direction, but the "shape" of the adjusted heights is also significantly affected. Another observation is that the two models yield very similar heights along the Swedish coast, which is only what could be expected considering the previous results; cf. Section 2.3. It is thus concluded that the present modifications of Lambeck's model yield significant height improvements over central Sweden. Along the coast, on the other hand, the differences are more or less negligible.

4.3 Comparison of RH 2000 with other height systems

With all components of the system definition being fixed, it is now possible to make the final adjustment of the Baltic Levelling Ring. It is not the purpose of the present document to describe the computation of geopotential numbers, gross error detection, adjustment, etc., in detail. Let us just mention a few basic facts concerning the RH 2000 adjustment.

All levelling data from the whole Baltic Levelling Ring was included in the adjustment. It should be stressed that only levelling observations were utilised. In the first step, the levelled height differences were converted to geopotential differences by multiplication with gravity (Heiskanen and Moritz 1967). A least squares adjustment was then made of the geopotential differences between a total of 7 400 nodal points, of which 5132 are Swedish. The national data sets in the Baltic Levelling Ring were given the weights determined by Karsten Engsager on behalf of NKG. The Swedish aposteriori standard error of unit weight became approximately 1 mm/√km. The estimated standard errors with respect to the NAP are approximately 2 cm in Sweden. The uncertainty due to the poor land uplift knowledge around Amsterdam (discussed in Sect. 4.1) is thus within the estimated standard errors, which is reassuring. In case the standard errors are transformed so that they refer to a fixed station in

Sweden, for instance Gävle, they become smaller than 1 cm for the whole country, increasing approximately as the square root of the distance. The relative standard errors inside Sweden are thus below 1 cm, which is important in practice.

The results from this adjustment constitute the new Swedish height system RH 2000. In this section we take a look at how well the final product compare with three other height systems, the old Swedish one, EVRF 2000 and the modern Danish height system. The goal is both to investigate the properties of the new height system and to learn something about the final land uplift model NKG2005LU. The comparison with the old Swedish system RH 70 is interesting since it might provide us with a clue concerning the magnitude of the levelling errors that did not go into the land uplift model. As has been discussed many times above, we believe it crucial to use a smoothing uplift model that filters a comparatively large portion of the observation errors, which is the main reason for that a smoothed land uplift model having been chosen. The comparison with EVRF 2000 is included mainly as an illustration of the differences over Sweden, caused by the significantly different land uplift epochs; cf. Section 1.2.

We start with the old Swedish height system RH 70, which is the system (frame) that resulted from the second precise levelling in Sweden; see for instance Ekman (1998). As stated in the last section, RH 2000 is defined as a realisation of EVRS, which implies that the NAP reference level has been used to fix the datum, and that it uses a zero system for the permanent tide. It should further be remembered that the land uplift epoch is 2000.0. The old system RH 70, on the other hand, utilises a non-tidal treatment of the permanent tide and has the uplift epoch 1970.0 (see Ekman 1998). What is common between the two systems is that RH 70 is defined using the same NAP zero level and that normal heights are utilised in both cases. To be fair, we should thus take care of the following two effects when RH 70 and RH 2000 are compared:

- the land uplift between 1970.0 and 2000.0 and
- the difference in permanent tide system (non-tidal and zero).

However, before these corrections are considered, let us take a look at how the heights of the two systems differ as they are. The height differences are illustrated in Fig. 4.6 and the corresponding statistics is given in Table 4.2. The most notable thing is that the land uplift dominates totally, but it is possible to discern also other effects of more short-wavelength nature; see for instance the "mountain" in Karlstad ($\phi = 59.5^{\circ}, \lambda = 13.5^{\circ}$).

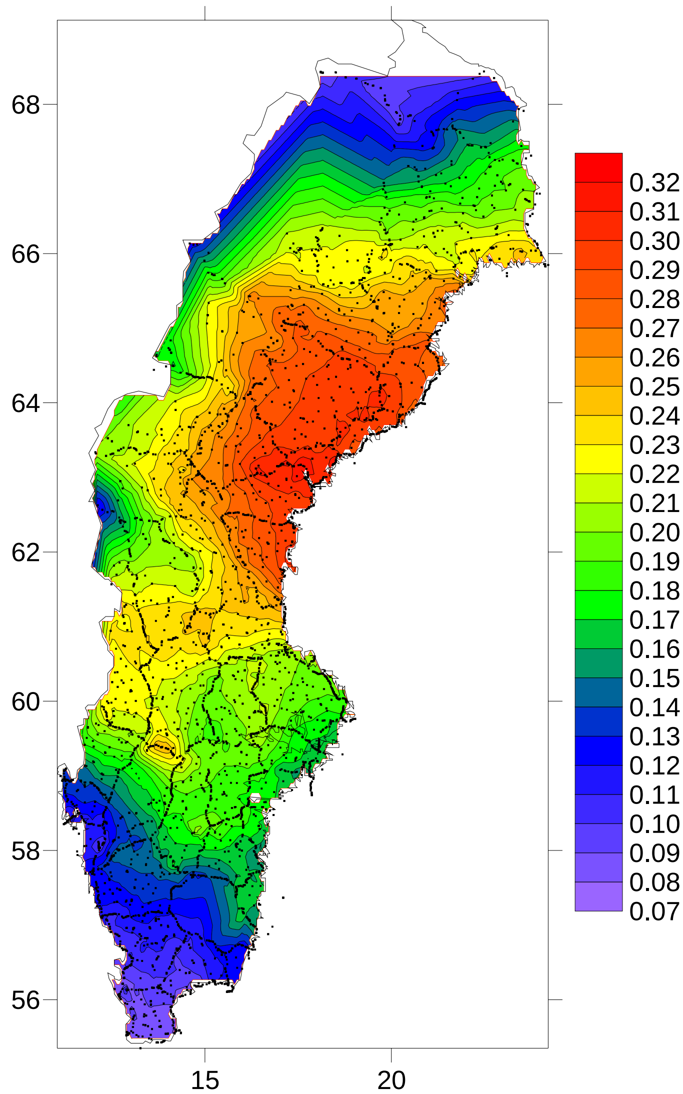

Figure 4.6: Difference between RH 2000 computed with NKG2005LU (smoothed inverse distance model with minimum -2.00 mm/year) and RH70. Unit: m.

Table 4.2: Statistics for the difference between RH 2000 computed with NKG2005LU (smoothed inverse distance model with minimum -2.00 mm/year) and RH 70. Unit: m.

Corrections	#	Min	Max	Mean	StdDev	RMS
No corr.	4751	0.072	0.311	0.191	0.060	0.207
Land uplift and permanent tide	4751	-0.155	0.048	-0.035	0.031	0.047

Let us now compare the two systems with corrections applied for the treatment of the permanent tide and the land uplift. The former is straightforward and is given by for instance Ekman (1989),

$$\Delta H_{zero} - \Delta H_{non-tidal} = 0.296 \cdot (\gamma - 1)\left(\sin^2\phi - \sin^2\phi_{NAP}\right) \qquad [\mathrm{m}] \qquad (4.1)$$

where ΔH_{zero} and $\Delta H_{non-tidal}$ denote height differences with respect to the NAP (i.e. $\Delta H = H - H_{NAP}$) and $\gamma = 0.8$ for RH 70 (Ekman ibid.). It should be noticed that only the height differences with respect to NAP should be corrected. As discussed above in Section 3.3, this can be motivated by the same NAP value having been applied during the years, independently of permanent tide system.

The land uplift correction is more complicated due to the way RH 70 was connected to NAP. We first point out that no correction should be applied to NAP and that only the height differences with respect to the NAP should be corrected. This follows from NAP being fixed in the European systems/frames; see Section 3.3 for details. To be able to see how the land uplift correction should be applied, it is important to explain how the connection to NAP was accomplished for RH 70, and the way the land uplift was treated. The adjustment was made with the height of a benchmark in Helsingborg as fixed. The approximate epoch of this benchmark is 1950.0 (Ekman 1994). All levelled height differences, as well as the Helsingborg benchmark, were reduced to the epoch 1970.0 using a land uplift model. The latter was constructed from the first and second Swedish precise levellings and 11 mareographs (Ekman 1998). Since no difference was considered between the apparent and levelled land uplifts (i.e. the eustatic sea level rise was neglected), a land uplift value very close to zero was applied for the Helsingborg benchmark. The value 0.06 mm/year was derived, which gives ≈ 1.2 mm *height* correction in Helsingborg. Practically, this means that the uplift between NAP and Helsingborg was not corrected between 1950.0 and 1970.0. From the above reasoning, it can be deduced that the following corrections should be applied to mediate between the two height systems:

$$H_{RH\,2000} - H_{RH\,70} = 0.296 \cdot (\gamma - 1)\left(\sin^2 \phi - \sin^2 \phi_{NAP}\right) + \left(\Delta \dot{H}_a + \Delta \dot{H}_e\right)(2000 - 1970) + \left(\Delta \dot{H}_a + \Delta \dot{H}_e\right)_{\text{Helsingborg}} (1970 - 1950) \quad [\text{m}] \tag{4.2}$$

where $\Delta \dot{H}_a$ and $\Delta \dot{H}_e$ are the apparent land uplift and eustatic sea-level rise differences between the point in question and NAP, respectively. Using the RH 2000 land uplift model the uplift difference between NAP and Helsingborg is approximately 2 mm, which implies that the last term in Eq. (4.2) amounts to an additional negative correction of 40 mm. It is acknowledged that the epoch of the early NAP connection is uncertain, and that other factors very likely are involved that could result in large systematic differences between the two systems. Take for instance the 1950 connection between NAP and Helsingborg: at best its standard error is a few centimetres. It has also been reported that the reference benchmark No. 000A2530/13600 (see Section 3.3), which is the only remnant of the old NAP tide gauge, has moved locally as much as 20 mm; cf. Mäkinen (2004). However, it should be noticed that our main goal here is to take all *known* effects into account to see how good the agreement becomes. As long as the systematic difference between RH 2000 and RH 70 can be explained within a number of centimetres, everything must be considered to be in order. Furthermore, the most interesting thing is how large the internal differences are within Sweden.

The differences between RH 2000 and RH 70 with the corrections in Eq. (4.2) applied are illustrated in Fig. 4.7 and the statistics can be found in Table 4.2. As can be seen, the mean (systematic) difference between RH 2000 and RH 70 is -35 mm, which is definitely acceptable; cf. the discussion in the last paragraph. It should also be noted that the difference is almost zero in the Southern part of Sweden, but the differences become larger the further ones moves to the north. This slope has been observed also in other circumstances, for instance when we have used the three precise levellings to estimate the land uplift in absence of other information.

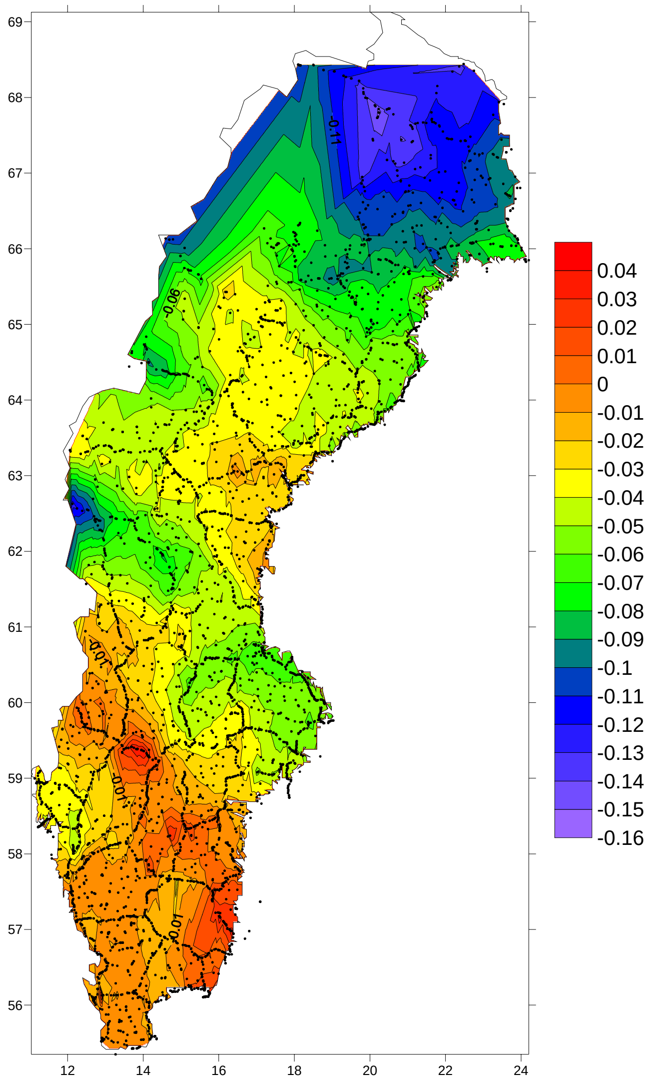

Figure 4.7: Difference between RH 2000 computed with NKG2005LU (smoothed inverse distance model with minimum -2.00 mm/year) and RH70. Corrections applied for the different land uplift epochs and permanent tide systems. Unit: m.

Thus, if corrections are applied for all known differences, the mean for the difference between RH 2000 and RH 70 is clearly acceptable, but RH 70 is affected by some kind of systematic error that increases to the north. Since the reliability and accuracy can be considered to be significantly higher for RH 2000 compared to RH 70, it seems reasonable to assume that the differences in Fig. 4.7 mainly reflects errors in the latter. Of course, no reference system is ever perfect, but considering the much more dense and homogeneous network (see Fig. 2.3), it seems justified to regard RH 2000 as the more accurate of the two. It can be seen in Fig. 4.7 that the internal differences between the two system show strong spatial correlations; see for instance the large deviations to the north, but some more local effects can also be spotted. It should further be noted that the standard deviation for the differences (see Table 4.2) is 3.1 cm, which is more or less what could be expected. Otherwise we believe that Fig. 4.7 speaks pretty much for itself. It shows clearly how the two systems (frames) differ from each other.

It should be pointed out that RH 70 was originally realised only through the levelling lines of the second precise levelling (see Fig 2.3). However, during the work with the third levelling, new heights were determined inside the RH 70 loops, which led to so-called RHB 70 heights. This explains that we have been able to compare RH 2000 and RH 70 inside the loops. It should be noticed, though, that the RHB 70 heights has been computed by fixing the original benchmarks to their RH 70 values, using observations from the third precise levelling. This means that most of the discrepancies that can be seen in Fig. 4.7 stem from the system differences along the original RH 70 lines. The values in between are mainly interpolated.

We turn now to the question what the comparison of RH 2000 and RH 70 might tell us about the land uplift model, i.e. NKG2005LU (smoothing inverse distance model with -2.0 minimum). First, the mean value shows that the datum difference caused by the two NAP connections is small. This means that the system resulting from choosing the uplift value to -2.0 mm/year in NAP is acceptable as far as the relation to RH 70 is concerned. It is more difficult to judge whether the line between observation errors and land uplift has been correctly drawn. We believe that the rather strong smoothing used for NKG2005LU is needed, and that the discrepancies that can be seen in Fig. 4.7 are mostly observation errors. Of course, it is impossible to prove this conclusively and one can always argue that we have drawn the line incorrectly. However, as long as we start from Vestöl’s point values, which imply a considerable smoothing in themselves, the adjusted heights are affected comparatively little by the choice of an exact or a smoothing interpolation (e.g. Fig. 3.14). It should also be pointed out that Vestöl (2005) has detected several gross errors in the RH 70 lines (see Section 2.2), which explains why some of the most notable features in Fig. 4.7 have not affected the land uplift model in a more pronounced way.

Let us now take a look at how RH 2000 computed using NKG2005LU agrees with the latest realisation of EVRS, i.e. EVRF 2000; see Ihde and Augath (2001). The differences between RH 2000 and EVRF 2000 at a number of nodal benchmarks are illustrated in Fig. 4.8.

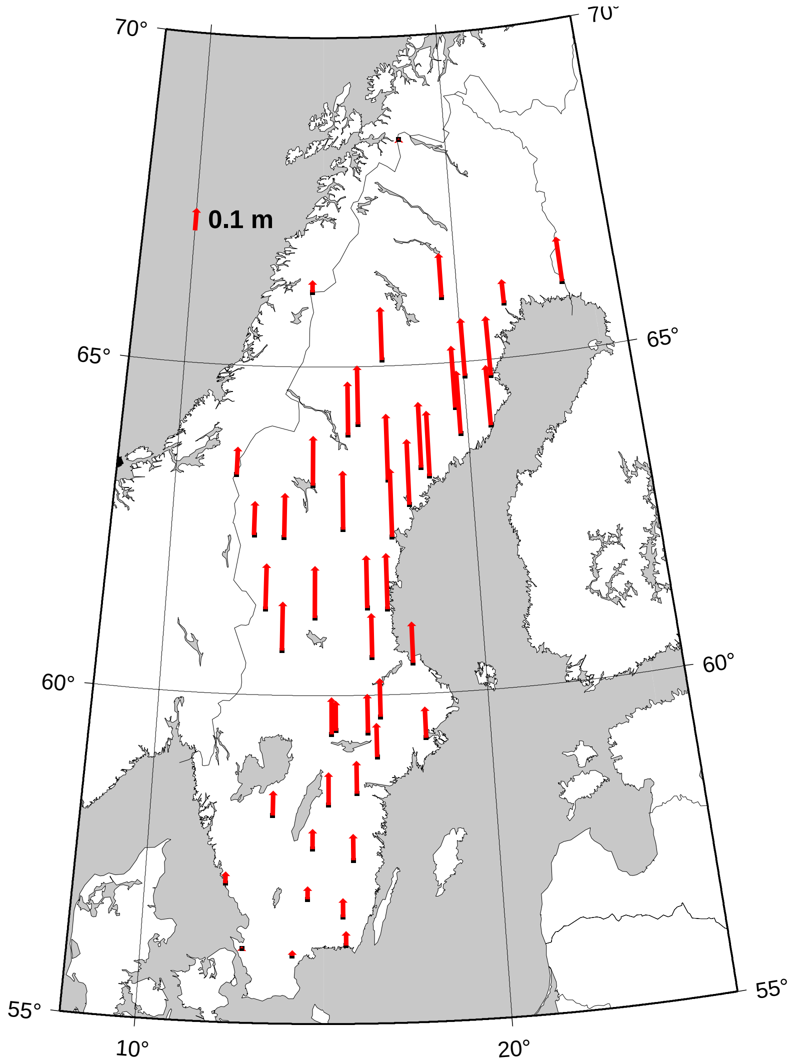

Figure 4.8: Difference between RH 2000 and EVRF 2000 at a number of Swedish nodal benchmarks.

The most notable feature in Fig. 4.8 is the magnitude of the discrepancies, which are mainly caused by the use of different land uplift epochs. As discussed in Section 1.2, the levelling observations in question were reduced to the epoch 1960.0 before delivery to the UELN database, while RH 2000 utilises the epoch 2000.0. It is thus clear that the large vectors in Fig. 4.8 mainly reflect the land uplift during 40 years. Another observation is that the two systems (frames) only differ approximately 1 to 2 cm in Helsingborg in southern Sweden, which is a somewhat surprising considering the land uplift difference between Helsingborg and Amsterdam (NAP). However, the question why RH 2000 and EVRF 2000 do not differ more in the southern part of Sweden will not be investigated further at the present point, but is left for the future. The main reason for presenting the comparison of RH 2000 and EVRF 2000 here is to emphasise that the two reference frames differ significantly in Sweden due to the widely separated land uplift epochs. This should always be kept in mind

Let us finish this section by noting that a direct comparison of the RH 2000 adjustment of the Baltic Levelling Ring with the modern Danish height system (DVR 90) shows that the two systems differ approximately 2 cm at Själland in the eastern part of Denmark. Since the Danish system has not been established using the NAP (Schmidt 2000), the result is very encouraging: The difference is sufficiently small to be neglected in almost all practical circumstances.

4.4 Mean Sea Level (MSL) in RH 2000

Above we have treated the choice of system definition and land uplift model for RH 2000. It is important to notice that these choices have not been performed blindly. Naturally we have studied the resulting RH 2000 heights and compared them both with the Mean Sea Level (MSL) along the Swedish coast and with other height systems. For instance, we would not have accepted NAP as zero level in case the resulting MSL was completely inappropriate for the Baltic Sea. It is the main purpose of this section to study the MSL in RH 2000 at the Swedish coasts.

The MSL in RH 2000 at the epoch 2000.0 for 4 Swedish mareographs is illustrated in Fig. 4.9. The computation was made as a linear regression using 90-120 years of observations lasting until 2001. The data (yearly mean values) were obtained from the Swedish Meteorological and Hydrological Institute (SMHI). No corrections were applied to the sea level observations.

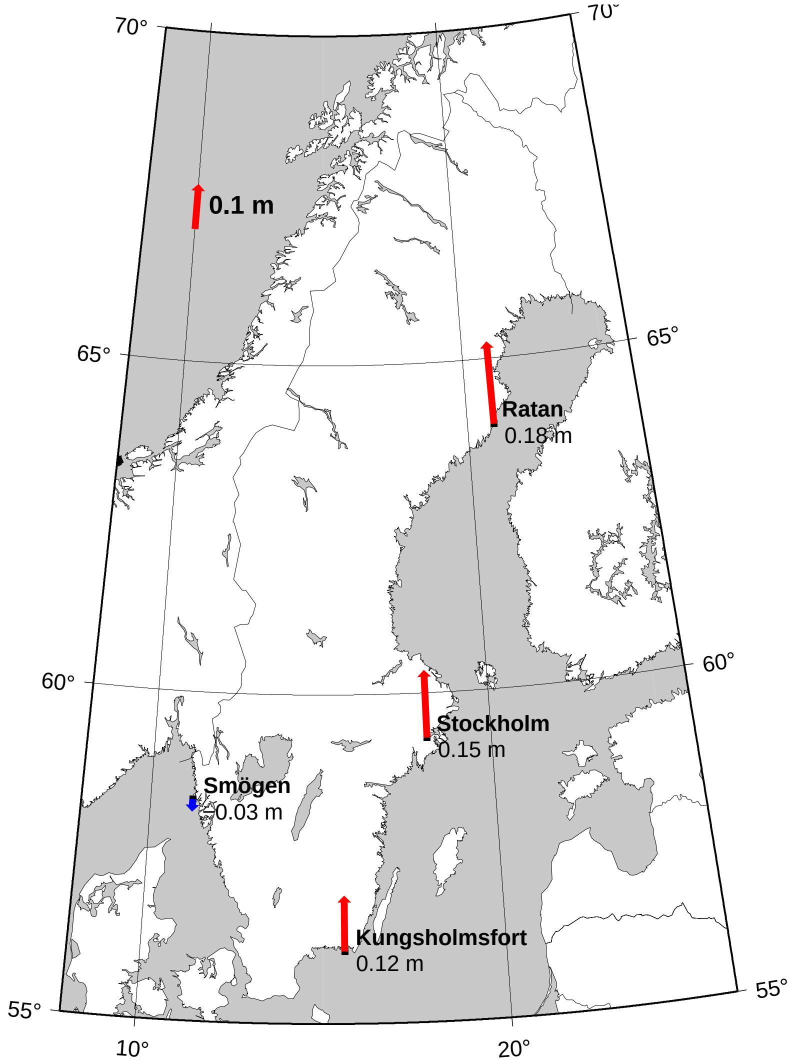

Figure 4.9: Mean Sea Level (MSL) in RH 2000 at 4 tide gauges. Epoch: 2000.0.

By studying Fig. 4.9, it can be seen that the MSL at epoch 2000.0 is reasonably close to zero in the western parts of Sweden and that the magnitude increases the further north one moves in the Baltic Sea. The main deviation is due to the sea surface topography and the fact that a zero permanent tide system is used for RH 2000; see Ekman and Mäkinen (1996b). Due to the mentioned effects, it is not possible to choose a zero level for RH 2000 so that the MSL becomes zero everywhere. Seen in this light, the obtained result seems good enough. The MSL is almost zero at the west coast and increases the further one moves into the Baltic Sea, which is appropriate considering the sea surface topography (Ekman and Mäkinen 1996b). It should also be noticed that what is discussed here is the MSL at the epoch 2000.0. As times moves on, the sea level will reduce due to the land uplift. This means that the MSL of the northern parts of the Baltic Sea will become smaller and become even closer to zero in RH 2000. It is concluded that the choice of NAP as zero level yields a system with heights agreeing reasonably well with the MSL at the Swedish coasts.

5. Summary and discussion

To be able to compute the new Swedish gravity-related height system RH 2000 using the observations of the third precise levelling in Sweden, the system (datum) needs to be defined by specifying

- which land uplift model to use,
- to which reference epoch the observations are to be reduced,
- how the zero level should be fixed,
- what type of heights that should be preferred and
- which system that is to be applied for the permanent tide.

It is admittedly somewhat unusual to include the land uplift model and epoch as part of the system definition, but we believe that this is motivated by the importance of the corresponding corrections in the Fennoscandian area.

The main purpose of this report has been to present the work conducted at the National Land Survey of Sweden (Lantmäteriet) to derive a land uplift model for the computation of the new Swedish height system RH 2000 and to discuss the choice of system definition. This work has been conducted in close cooperation with the other Nordic countries within the height determination group of the Nordic Geodetic Commission (NKG). Due to the acute need of a new height system in Sweden, Sweden was forced to finalise the project in the beginning of 2005. To obtain heights that agree as well as possible with other European countries, it was decided that RH 2000 should be defined as a Swedish realisation of the 2005 version of the EVRS (European Vertical Reference Frame). This implies (according to our interpretation) that the NAP is used to give the zero level, that normal heights are utilised and that a zero system is applied for the permanent tide. However, nothing is specified concerning kinematic (land uplift) corrections at the European level. The land uplift model and reference epoch therefore remained to be chosen on the national (Nordic) level. The epoch was chosen together with Finland and Norway to 2000.0.

The computation of a land uplift model for RH 2000 did not start from scratch. Instead it was decided to continue the work so far made within the NKG. The NKG situation at the end of 2004 was that two different models had been chosen for further consideration, namely the geophysical model of Lambeck et al. (1998) and the mathematical model of Vestøl (2005). The first model was tuned to 58 mareographs in the Nordic area (see Ekman 1996) and to ancient shore line observations. Other knowledge of the physical behaviour of the Earth is naturally also taken advantage of. The Vestøl (2005) model, on the other hand, is a purely mathematical construct, which has been estimated from almost all available observations using least squares collocation (e.g. Moritz 1980). Besides the 58 tide gauges just mentioned, Vestøl utilised 55 GPS-derived absolute uplift rates from Lidberg (2004) as well as repeated levelling in Sweden, Finland and Norway. A detailed discussion of the different observations has been included in Chapter 2. This information, i.e. the land uplift models of Vestøl (2005) and Lambeck et al. (1998), has been considered as the starting point in the present work. This means that no attempts have been made to estimate the best possible model from scratch. Instead the goal has been to modify and/or combine the two models in the best possible way.

It might be asked why the two models need to be modified in the first place. The answer is that none of them is sufficiently good for the present purpose. The main problem with Lambeck's model is that it fits poorly with many of the available observations, most notably with almost all GPS observations in the central parts of Sweden. In Chapter 2 it has been found that the magnitude of the deviations is 1 mm/year or more from Kiruna in the north to Jönköping in the south of Sweden. Naturally, Vestøl's model fits much better to the observations, but it has other drawbacks. First, Vestøl (2005) uses least squares collocation to estimate the uplift in the observation points only. An independent gridding algorithm is then taken advantage of to produce a regular grid. Unfortunately, this two-step procedure results in a bad behaviour of the model in those areas where only a few observations are available and at the outer borders. The bad behaviour consists of staircase cylinders, which can be clearly spotted in for instance Fig. 2.10. Another problem with Vestøl's model is that it does not cover a sufficiently large area. To be able to connect to the NAP, the Nordic levelling networks had to be augmented with the networks of the the Netherlands, northern Germany, Poland and the Baltic countries, forming what has been named the Baltic Levelling Ring. Other purposes of defining the Baltic Levelling Ring is to be able to relate the Nordic height systems to each other and to be able to check loops around the Baltic Sea and Gulf of Bothnia. To close the loops, however, other information besides levelling is required; cf. below. The final RH 2000 adjustment was to be made of the whole Baltic Levelling Ring network (illustrated in Fig. 1.1), which naturally requires that the land uplift model has to cover the same area. This is not the case for Vestøl's model. The final problem with Vestøl's model is that the contour lines are curvy, showing a clear zigzag behaviour. This indicates that the model might not have been sufficiently smoothed. Due to the rather high rigidity of the crust, the land uplift cannot have any funny shape, but is bound to be comparatively smooth; cf. the geophysical model of Lambeck illustrated in Figs. 2.14 and 2.15. Very likely, the small detail that can be discerned in Vestøl's model is caused by observation errors. By some additional filtering, it is believed that the influence of the observation errors can be diminished and that a more realistic model can be obtained.

The most important part of the work has been to investigate different ways to combine the models of Vestöl and Lambeck. The main strategy has been to start from Vestöl's model as defined in the observation points and then use a remove-compute-restore technique with respect to Lambeck's model. The compute step here means the application of an interpolation method. However, as it was not known from the start that Vestöl used an independent gridding algorithm, many tests were made starting from Vestöl's grid values. The results from these tests have also been summarised, mainly to provide a background that motivate some of the later developments. The main result here has been that it is not suitable to start from Vestöl's grid model. It is more or less impossible to get rid of the disturbing "cylinders". Consequently, the only viable alternative is to consider Vestöl's model as defined in the observation points. Four different interpolation methods have been investigated, namely

- Exact inverse distance interpolation (Bjerhammar's deterministic method) with power parameter 3.
- Smoothing inverse distance interpolation (Bjerhammar's deterministic method) with power parameter 3 and smoothing chosen so that the smoothest possible model is obtained at the same time as the surface does not deviate more from the

observations in an RMS sense than the corresponding standard errors.

- Kriging as implemented in SURFER 8 (Golden Software Inc. 2005) using a Gaussian variogram with Nugget effect. The parameters for the former part were found by analysing the spatial correlations for the field in question, i.e. for the differences between Vestöl's and Lambeck's models. The Nugget effect, which reflects the magnitude of the observation errors, was somewhat arbitrarily chosen to 0.2 mm/year for all observations. It is not possible in SURFER 8 to use different weights for different observations.
- Least squares collocation with individual weights for each observation as implemented in GEOGRID (Forsberg 2004). A 2nd order Markov covariance function is assumed with the parameters chosen by analysing the field. The estimated standard errors for the model of Vestöl (2005) were utilised for the weighting.

The four interpolation schemes have been analysed in great detail. The smoothing inverse distance method was finally singled as the most suitable technique for the present purpose. The basic requirements put on the interpolation method are that the resulting model should

- be as smooth as possible at the same time as it does not differ more in an RMS sense from the tide gauge and GPS observations than their respective standard errors. The model should thus "look" realistic, which is the requirement used to ensure that it is reasonably meaningful from a physical point of view.
- approach Lambeck's model in a good way outside the high quality observations in the central Nordic area. It should also behave well in areas with no data.

The last requirement rules out many interpolation methods that are excellent in areas with observations, but which behave more or less arbitrarily otherwise. It is a clear advantage of the smoothing inverse distance method that it is robust at the border to, and immediately outside, the area with observations. Both the collocation and Kriging solutions (with the chosen parameters) have a tendency to oscillate a little bit too much in such areas. It should be pointed out that all interpolation methods can be tuned in different ways and that the above conclusions only refer to the methods with the chosen parameters. For instance, Kriging with another type of variogram naturally behaves differently.

The main result of Chapter 3 has thus been the choice of interpolation and extrapolation method (smoothing inverse distance method). To see that everything is in order with the corresponding Baltic Levelling Ring adjustments, the closing errors around the Baltic Sea and the Gulf of Bothnia have also been studied. This requires other information besides levelling, such as an oceanographic estimate of the difference in Mean Sea Level (MSL) at two benchmarks. The closing errors are smaller than expected, but it is not possible to conclude that one interpolation technique is better than the other in this respect. Everything have been found to be in order with the adjustment, though,

The most important topics of Chapter 4 have been to discuss how the land uplift (sinking) in the NAP should be treated, to choose final land uplift model and to investigate the adjusted heights resulting from the chosen model and system definition. It is concluded that the 2005 version of EVRS has to be interpreted as being realised through a fixed height of the NAP benchmark, which is independent of both kinematic movements and of the permanent tide system. Due to the fact that the smoothing inverse distance model (Chapter 3) yields a land sinking that is a little too large in NAP, the model is changed accordingly. This is accomplished by setting all apparent land uplift values below -2 mm/year equal to this value (-2 mm/year), which is the same technique as was utilised for the digitised model of Lambeck. Before this new minimum was applied, it was investigated how much the adjusted heights depend on the choice. It has been concluded that a change of the minimum value of 0.5 mm/year yields a 1 cm systematic difference over Sweden. Consequently, since it seems reasonable to assume that the standard error of the NAP uplift is 0.5 mm/year or a little higher, a systematic shift with standard error 1 to 2 cm must be expected over Sweden. Considering that the standard errors relative to the NAP are considerably larger (approximately 2 cm), it is concluded that the adopted strategy is sufficiently good for the purpose.

After the choice of final uplift model (called NKG2005LU), the resulting RH 2000 heights have been investigated in a number of ways. It should be emphasised that it has not been the purpose of the present report to present any details concerning the practical observations or the actual computation. It suffices to say that the Baltic Levelling Ring network, of which the RH 2000 network is a subset, has been adjusted after conversion of the levelled height differences to geopotential numbers utilising interpolated gravity values. Only levelling observations have been included in the adjustment.

Let us turn now to the investigations of the adjusted RH 2000 heights referred to above. First, RH 2000 has been compared to the old Swedish system RH 70. Not surprisingly, the main difference is due to the land uplift during 30 years. It has also been investigated how well the systems agree after corrections have been applied of all known effects (land uplift and permanent tide system). The RMS of the differences reduces from 20.7 to 4.7 cm when the corrections are applied. Nevertheless, large differences remain in some areas, for instance in the northernmost parts of Sweden, where the deviations are larger than 1 dm; see Fig. 4.7 for details. Due to the comparatively high redundancy of RH 2000, it is likely that the largest errors are in RH 70. We believe that the comparison between RH 70 and RH 2000 is interesting insofar as RH 70 is quite a typical height system. It is not meaningful to expect 1 cm agreement in the RMS sense between geometric (GPS/levelling) and gravimetric geoid heights (height anomalies) for such a system.

The next height comparison has been to study the difference between RH 2000 and EVRF 2000. Also in this case, the deviations are large, mainly due to the phenomena of land uplift. The land uplift epoch for the Nordic block in EVRF 2000 is 1960.0, which means that 40 years of land uplift show up. The main purpose of the comparison has been simply to illustrate the large difference; see Fig. 4.8. Finally, the RH 2000 heights have been compared with the modern Danish height system DVR 90. Even though our colleagues in the west have not defined their system using the NAP (they have utilised a number of tide gauges along the Danish coast), RH 2000 and DVR 90 only differ by approximately 2 cm in Själland (eastern part of Denmark).

It has further been studied how well the final RH 2000 heights agree with the Mean Sea Level (MSL) at the epoch 2000.0 along the Swedish coast. The MSL (epoch 2000) is slightly below zero at the west coast of Sweden and increases the further one moves into the Baltic Sea. In the northern part of the Baltic Bay it is approximately 18 cm. This increase depends on the sea surface topography; see Ekman and Mäkinen (1996b). Since the postglacial land uplift will reduce the MSL in RH 2000 as time passes, it can be concluded that the NAP zero level gives a reasonable fit to the MSL.

The main part of the work presented in this report consists of the computation of a new land uplift model, the other aspects of the system definition being either decided on a European level (NAP, zero permanent tide, normal heights) or in Nordic collaboration (the land uplift reference epoch 2000.0). It might be thought that the path followed by the authors is somewhat strange. Why modify the models of Vestöl and Lambeck instead of constructing a brand new model from scratch, which fulfils all the requirements directly? The answer is partly that the time schedule did not allow for such far-reaching excursions. Besides, it is believed that the final model, i.e. NKG2005LU, is about as good as a land uplift model can be at the present time, considering the available observations and knowledge. The model describes a smooth, realistically looking field that agrees well with the observations. In the central parts of the area, the model is more or less a smoothed version of Vestöl's mathematical model while Lambeck's geophysical model is utilised more and more the farther one moves from the uplift center. In this way, the main drawbacks of the two models are avoided and a reasonable compromise is obtained. The model NKG2005LU (RH 2000 LU) is illustrated in black and white and below.

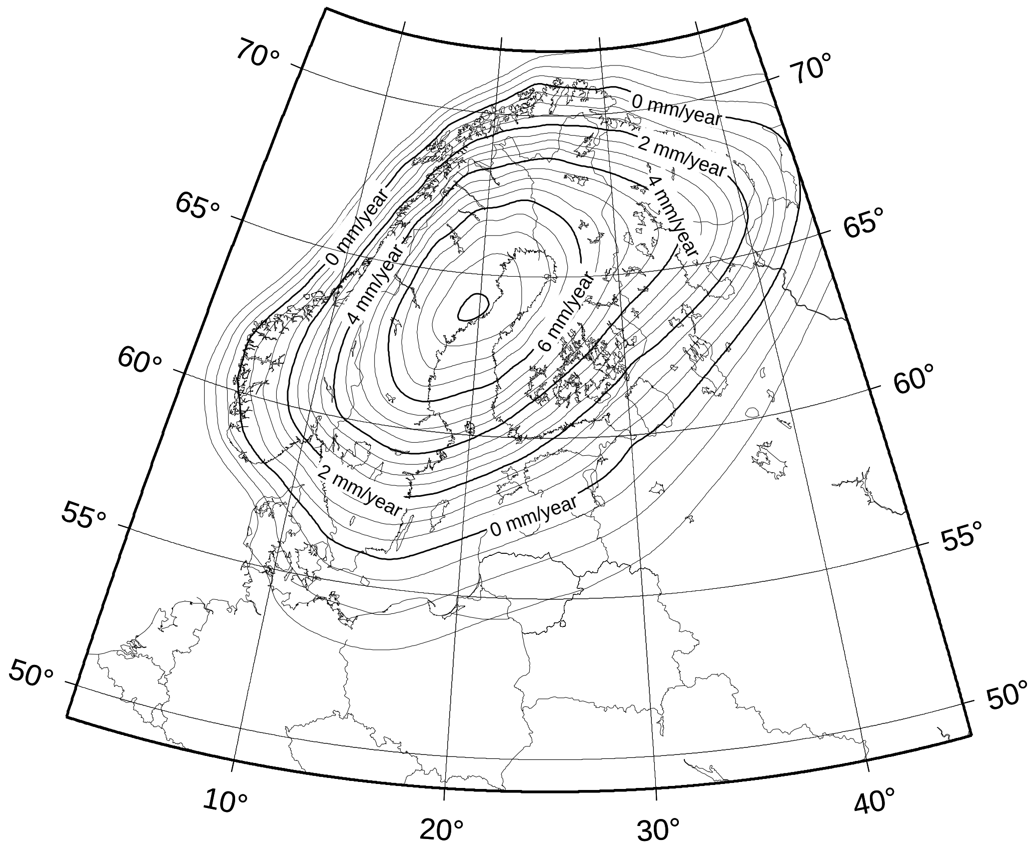

Figure 5.1: Apparent uplift for the final land uplift model NKG2005LU (RH 2000 LU).

As remarked in Sect. 1.5, the full text of this report was written in 2005, but is not published until now (2007). No substantial changes have been made to the 2005 version except as noted in Sect. 1.5. Since 2005, the land uplift model has been accepted as a Nordic model by the NKG and the RH 2000 adjustment has been taken as giving the official result of the Baltic Levelling Ring project. Furthermore, Olav Vestøl has published an improved version of his land uplift model in Journal of Geodesy (Vestøl 2006). The European Vertical Reference System is also about to change so that a number of stations will be used instead of NAP to give the zero level, and so that the observations are reduced to a common reference epoch by means of a land uplift model; see Ihde et al. (2006).

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