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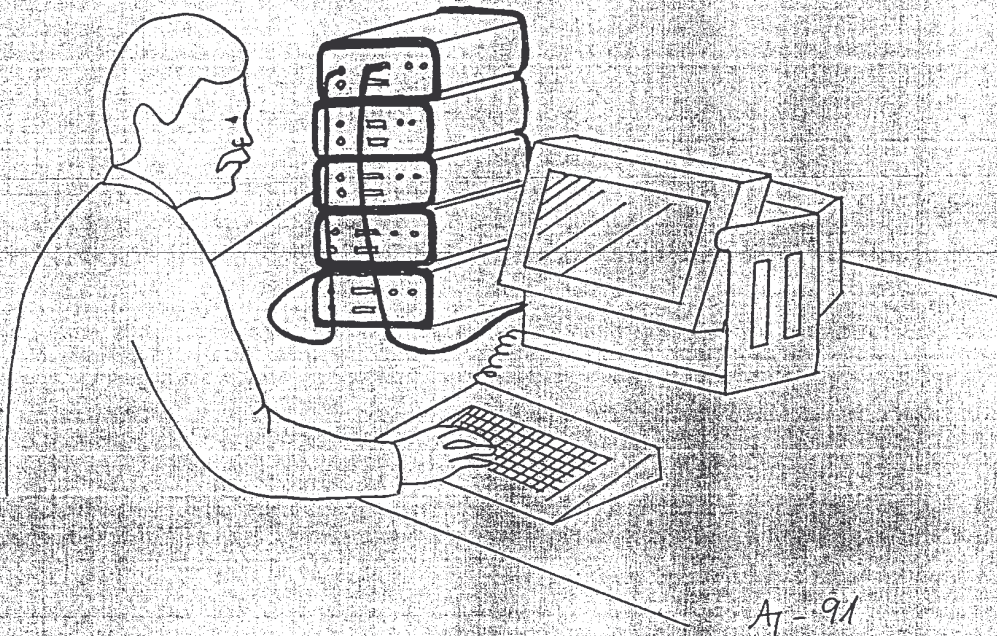
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GPS Computations and Analyses for Geodetic Control Networks

by

Lotti Jivall



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GPS Computations and Analyses for Geodetic Control Networks

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PREFACE

At the geodetic divisions of the National Land Survey of Sweden we aim at producing a complete guide for geodetic control surveying with GPS.

The part of the guide dealing with the planning of GPS networks has been issued (in Swedish) by Lithén & Persson (1991).

The report presented here deals with the computations and analyses of GPS data for control surveying. Although guide-lines and criteria are derived from experiences in Sweden it is believed that the contents might be of interest to geodesists working with similar problems in other countries as well.

The author would like to express her thanks to all those GPS users who have contributed with valuable remarks and comments.

Finally it should be mentioned that this report is a translation (with a few amendments) of the Swedish report "GPS-beräkning för stornät" (LMV-rapport 1991:18), the translation being made by my colleague Martin Ekman.

Lotti Jivall

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1. INTRODUCTION

During the latest years the GPS technique has become very important in geodetic control surveying. In Sweden such works are carried out by the National Land Survey of Sweden, and also by other surveying organizations. In 1985-88 some trial measurements and pilot projects were performed in order to establish measuring methods for various applications. In 1989 a more regular production of geodetic control networks was initiated. Because of this, a need for guide-lines for GPS measurements has arisen. An attempt at guide-lines for the design of areal GPS networks is presented (in Swedish) in Lithén & Persson (1991). The present report contains a corresponding attempt at guide-lines for the computation and analysis of GPS data.

The report is restricted to horizontal control surveying with single frequency measurements, the length of the baselines ranging from some hundred meters to tens of kilometers. This is the dominating type of GPS measurements in Sweden today. By the concept of control network is meant a geodetic network that is to serve many different purposes, also future purposes unknown today. The accuracy is to be as high as possible using ordinary methods. Thus there is no classification into different accuracy standards.

Guide-lines and criteria have been determined with respect to the capability of the present technique. The basis has been computations of control networks made by the National Land Survey during 1990 and 1991.

At the National Land Survey the following computation systems are used for the production of GPS control networks. The adjustment of GPS observations is made in the PC program GPPS from the instrument manufacturer Ashtech Inc., U.S.A.; this is a baseline program founded on adjustment of double differences. The network adjustment is performed with the PC program GeoLab from GEOSurv Inc., Canada, and the transformations with our own programs.

Criteria for analysing adjustment of GPS observations is given both for GPPS (Ashtech's program) and TRIMVEC-PLUS (Trimble's program). The criteria for GPPS are determined with the versions 90.04 - 91.07 but are valid also for earlier versions. For TRIMVEC-PLUS the version C has been used. The guide-lines given are valid for both programs, and should be applicable also for other program systems based on the same fundamental principles. The criteria, however, might require slight modifications.

Controls between sessions through comparisons of double-measured baselines or computation of loop misclosures are, on the other hand, independent of the software used at an earlier stage.

Also the network adjustment is of a quite general kind. Here we suggest a standard weighting, both in a local system (northing, easting, up) and in geocentric cartesian coordinates (X, Y, Z).

Two methods are presented for the connection to a supreme control network: the method hitherto used by the National Land Survey, i.e. Helmert transformation of a freely adjusted network, and a method where the connection stations are held fixed in the network adjustment. From the latter method we do not have very much GPS experience.

2. OUTLINE OF THE COMPUTATIONAL PROCESS

As has already been mentioned, only horizontal control surveying will be treated in this report. The computational process for that might be divided into the following main steps; see also figure 1:

- 1. Computation of initial coordinates in WGS84 for a starting point in the network.** This computation is made by a transformation from national coordinate and height systems (in Sweden RT90 and RH70). Hence one should plan the measurements in such a way that at least one station with coordinates known in these national (or similar) systems is included in the measurements of the first day.
- 2. Adjustment of GPS observations.** This can be performed either for each baseline separately or as a common adjustment of several baselines (multi-station adjustment).
- 3. Controls within and between sessions.** Controls within sessions are made either by multi-station adjustment or by one network adjustment per session. Controls between sessions are made through comparisons of double-measured baselines or computations of loop misclosures.
- 4. Network adjustment.** The baseline components from step 2 are adjusted to form a free, unified network in an approximate WGS84.
- 5. Connection to a supreme network.** The coordinates from the network adjustment are transformed to horizontal coordinates (grid coordinates) in an approximate national system (in Sweden RT90). The actual connection to the coordinate system in question is made by means of a plane Helmert transformation.

Alternatively, this Helmert transformation is considered a check only, and the final connection is made with a network adjustment where the latitudes and longitudes in WGS84 of the connection stations are held fixed. The WGS84 coordinates are obtained by transformation and are, after the network adjustment, reverted into the original coordinate system by transformation. It is, thereby, important that the two transformations really are the inverses of each other.

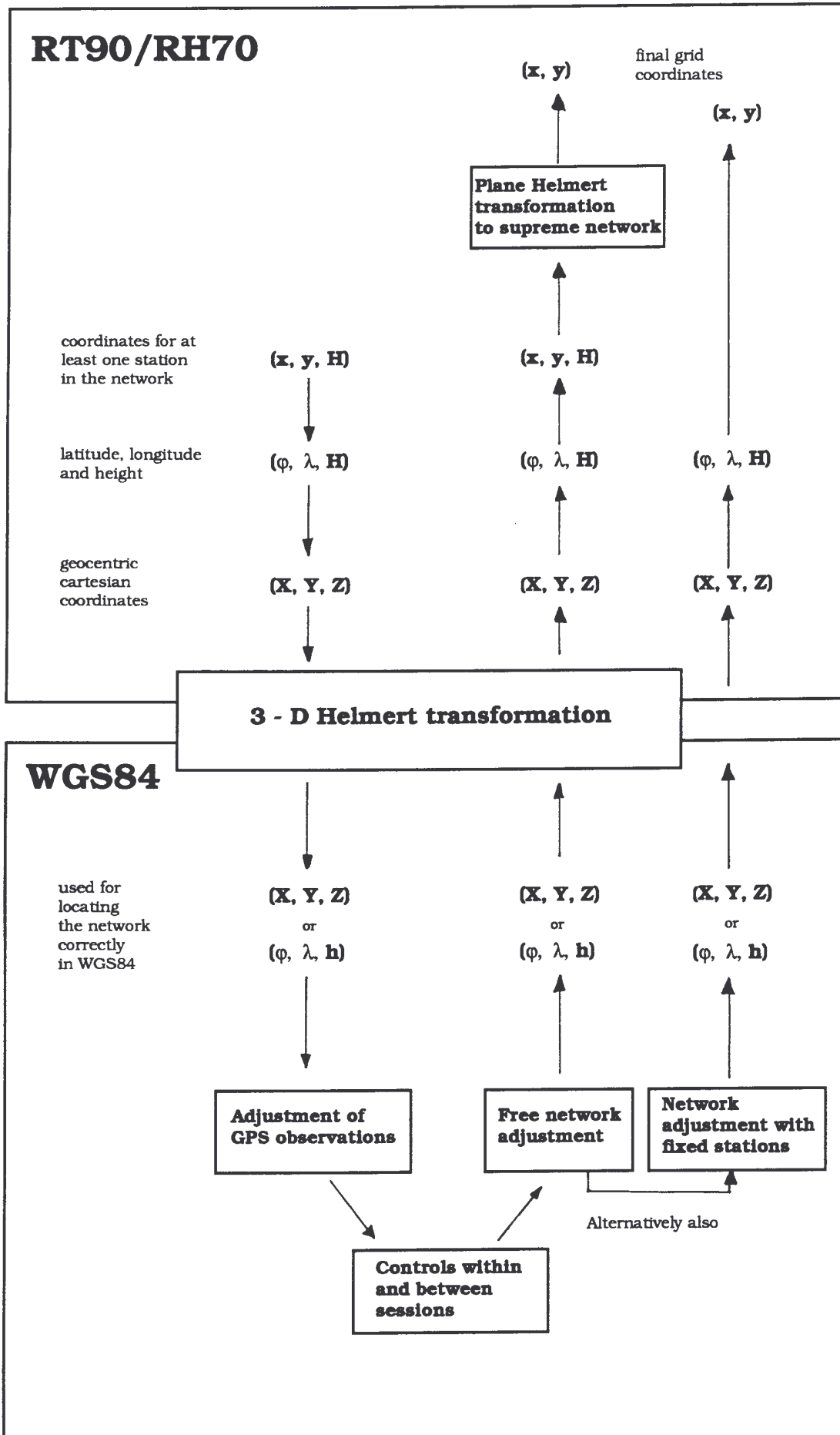


Figure 1. Process chart for computation of horizontal coordinates with GPS.

3. COORDINATE TRANSFORMATIONS IN CONNECTION WITH GPS

3.1 Transformations between WGS84 and RT90

GPS measurements and computations are carried out in the geodetic datum World Geodetic System 1984 (WGS84) which is a global datum with its origin located very close to the Earth's centre of gravity. The coordinates are expressed either as geocentric cartesian coordinates (X, Y, Z), the Z -axis practically coinciding with the Earth's rotational axis and the X -axis intersecting the equator at the Greenwich meridian (see figure 2), or as latitude, longitude and height above the ellipsoid (ϕ, λ, h). Transformations to and from geocentric coordinates are treated in appendix A; see also e.g. Leick (1990).

The ellipsoidal parameters for WGS84 coincides for all practical purposes with the Geodetic Reference System 1980 (GRS 1980).

The WGS84 ellipsoid:

$$\begin{aligned} a &= 6\,378\,137 \text{ m} \\ 1/f &= 298.257\,223\,563 \end{aligned}$$

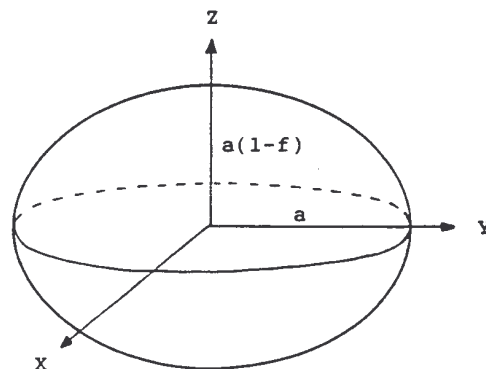


Figure 2. Geocentric cartesian coordinates and the ellipsoid.

A transformation formula, intended primarily for navigational purposes, has been established for the datum shift between WGS84 and the Swedish national coordinate system RT90 (Hedling & Reit 1990). The formula is founded on a three-dimensional Helmert transformation of 9 triangulation stations spread over the country. The WGS84 coordinates for these stations were obtained from two Scandinavian Doppler campaigns. X, Y, Z in RT90 were calculated from latitude and longitude in RT90 and from heights in the national height system RH70 (disregarding geoid heights!).

The Bursa-Wolf model for the 3-D Helmert transformation (7 parameter transformation) has been applied:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix} + (1 + \delta_s) R \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (1)$$

where

X, Y, Z	=	coordinates in the RT system
x, y, z	=	coordinates in the WGS system
$\Delta X, \Delta Y, \Delta Z$	=	translations
δ_s	=	scale correction
R	=	rotation matrix

The rotation matrix R may be written as a product of the rotation matrices R_x , R_y and R_z which perform, in turn, the rotations ω_x , ω_y and ω_z around the x -, y - and z -axes.

$$R = R_z R_y R_x = \begin{bmatrix} \cos\omega_z & \sin\omega_z & 0 \\ -\sin\omega_z & \cos\omega_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\omega_y & 0 & -\sin\omega_y \\ 0 & 1 & 0 \\ \sin\omega_y & 0 & \cos\omega_y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega_x & \sin\omega_x \\ 0 & -\sin\omega_x & \cos\omega_x \end{bmatrix} =$$

$$= \begin{bmatrix} \cos\omega_y \cos\omega_z & \cos\omega_x \sin\omega_z + \sin\omega_x \sin\omega_y \cos\omega_z & -\cos\omega_x \sin\omega_y \cos\omega_z + \sin\omega_x \sin\omega_z \\ -\cos\omega_y \sin\omega_z & \cos\omega_x \cos\omega_z - \sin\omega_x \sin\omega_y \sin\omega_z & \sin\omega_x \cos\omega_z + \cos\omega_x \sin\omega_y \sin\omega_z \\ \sin\omega_y & -\sin\omega_x \cos\omega_y & \cos\omega_x \cos\omega_y \end{bmatrix} \quad (2)$$

Please note how the rotations have been defined - sometimes they are defined with opposite signs.

From the Helmert transformation the following parameters were obtained:

$$\begin{aligned} \Delta X &= -424.3 \text{ m} \\ \Delta Y &= +80.5 \text{ m} \\ \Delta Z &= -613.1 \text{ m} \\ \omega_x &= -4.3965 \text{ seconds of arc} \\ \omega_y &= +1.9866 \text{ seconds of arc} \\ \omega_z &= -5.1846 \text{ seconds of arc} \\ \delta_s &= 0.0 \text{ mm/km} \end{aligned} \quad (3)$$

The standard error of the transformation is 2.4 m/coordinate and the residuals vary between 0.2 m and 1.0 m horizontally, and between 0.2 m and 7.2 m in height.

Formula (1) is to be used when transforming coordinates from WGS84 to RT90/RH70. If one wants to perform the reverse transformation (from RT90/RH70 to WGS84), requiring this to be consistent with the transformation from WGS84 to RT90/RH70, one must use the inverse of formula (1):

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{R^1}{(1 + \delta_s)} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} - \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix} \quad (4)$$

where $R^1 = R^T$, i.e. the transpose of the rotation matrix R .

If one does not have too high demands for consistency but can accept a slight change of the coordinates after transformations in both directions, one may exchange the rotation matrix R in formula (1) for a linearized form of (2):

$$R = \begin{bmatrix} 1 & \omega_z & -\omega_y \\ -\omega_z & 1 & \omega_x \\ \omega_y & -\omega_x & 1 \end{bmatrix} \quad (5)$$

Note that the rotations ω_x , ω_y , ω_z from (3) have to be converted to radians before they are inserted into formula (5).

For the reverse transformation (from RT90/RH70 to WGS84) one may then, instead of formula (4), apply formula (1) although with opposite signs of the transformation parameters (X, Y, Z and x, y, z will, in this case, exchange their meanings). This simplified procedure may cause differences of a few cm when going back and forth in the transformation chain.

When computing approximate WGS84 coordinates according to figure 1 one starts from national grid coordinates in RT90 which are transferred into latitude and longitude on the Bessel 1841 ellipsoid using the Transverse Mercator projection. Formulae for this map projection are given in appendix B; see also e.g. Krüger (1912).

The Bessel ellipsoid:

$$\begin{aligned} a &= 6\,377\,397.155 \text{ m} \\ 1/f &= 299.152\,812\,8 \end{aligned}$$

In combination with the national height in RH70, without geoid height, a transformation is then made to geocentric cartesian coordinates according to appendix A, to which formula (4) with (2) is applied. Possibly, one may instead use formula (1) with (5) and opposite signs of the parameters (3). The obtained geocentric WGS84 coordinates might then be transformed into latitude, longitude and height above the WGS84 ellipsoid according to appendix A.

3.2 Transformation of coordinate differences

In order to facilitate the analysis of GPS computations, which are often carried out in geocentric cartesian coordinates, one can transform residuals or other small coordinate differences to a local system (northing, easting, up; N, E, U). The relation between X, Y, Z and latitude, longitude and height may, in this case, be written somewhat approximately in the following way (cf. figure 3):

$$\begin{aligned} X &= (r + h) \cos \psi \cos \lambda \\ Y &= (r + h) \cos \psi \sin \lambda \\ Z &= (r + h) \sin \psi \end{aligned} \quad (6)$$

or

$$\begin{aligned} \psi &= \arcsin (Z / (X^2 + Y^2 + Z^2)^{1/2}) \\ \lambda &= \arctan (Y / X) \\ h &= (X^2 + Y^2 + Z^2)^{1/2} - r \end{aligned} \quad (7)$$

with

$$\tan \psi = b^2 / a^2 \tan \varphi = (1 - f)^2 \tan \varphi \quad (8)$$

where

$$\begin{aligned} X, Y, Z &= \text{geocentric cartesian coordinates} \\ r &= \text{radius vector} \\ h &= \text{height above the ellipsoid} \\ \psi &= \text{geocentric latitude} \\ \lambda &= \text{longitude} \\ \varphi &= \text{geodetic latitude} \end{aligned}$$

- a = semi-major axis
 b = semi-minor axis
 f = flattening

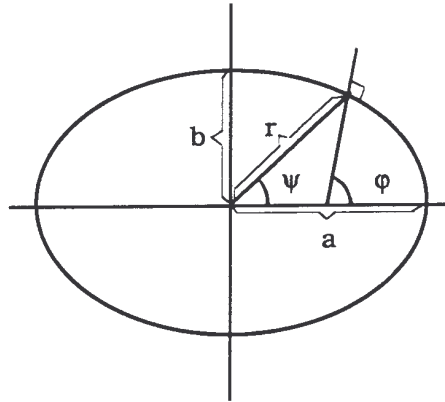


Figure 3. Relations between a, b, r, ψ and ϕ .

r varies with the latitude. In this context, however, we may neglect this fact and treat the Earth as spherical.

Differences in northing, easting and up (N, E, U) can be expressed as functions of latitude, longitude and height:

$$\Delta N = (r+h) \Delta \psi, \Delta E = (r+h) \cos \psi \Delta \lambda, \Delta U = \Delta h.$$

The relations between differences in X, Y, Z and differences in N, E, U have been derived by applying the law of propagation of errors, using Taylor expansions of (6) and (7) retaining first order terms only:

$$\begin{aligned}
 \Delta N &= -\sin \psi \cos \lambda \Delta X - \sin \psi \sin \lambda \Delta Y + \cos \psi \Delta Z \\
 \Delta E &= -\sin \lambda \Delta X + \cos \lambda \Delta Y \\
 \Delta U &= \cos \psi \cos \lambda \Delta X + \cos \psi \sin \lambda \Delta Y + \sin \psi \Delta Z
 \end{aligned} \tag{9}$$

Also the inverse relation may be of interest in certain cases, e.g. when studying the influence on the different components of an error in the antenna height:

$$\begin{aligned}
 \Delta X &= -\sin \psi \cos \lambda \Delta N - \sin \lambda \Delta E + \cos \psi \cos \lambda \Delta U \\
 \Delta Y &= -\sin \psi \sin \lambda \Delta N + \cos \lambda \Delta E + \cos \psi \sin \lambda \Delta U \\
 \Delta Z &= \cos \psi \Delta N + \sin \psi \Delta U
 \end{aligned} \tag{10}$$

For most applications it is sufficient to use latitude and longitude within 100 km (1°). This simplification might cause errors of the order of 5%. Accepting this, the geocentric latitude may be exchanged for the geodetic latitude (the maximum difference between them corresponding to a few tens of km).

4. ADJUSTMENT OF GPS OBSERVATIONS

4.1 General aspects

In this chapter we will deal with those steps in the computations where pseudorange and carrier phase measurements as well as satellite ephemerides are processed to produce baseline components (coordinate differences) between stations measured simultaneously with GPS.

There are many kinds of software available adopting different fundamental principles and more or less advanced models for the computations. The major differences concern the treatment of the carrier phase measurement: formation of differences (single, double or triple differences) or direct use of the raw phase observation, methods for determining cycle ambiguities, orbit determination, and the question if more than two stations can be processed in a simultaneous adjustment (multi-station program or baseline program).

There is software directly related to a certain receiver type as well as receiver-independent software often developed at universities. The latter ones are usually more flexible and contain more advanced models, but are often less adapted to ordinary computations. Via a standardized raw data format, RINEX (Receiver Independent Exchange Format), one normally has the possibility of switching to other program systems than the one developed by the manufacturer of the receivers one is using. Today, not all receivers (together with their postprocessing software) can produce data in RINEX format. Furthermore, some incompatibility between different RINEX formats may occur.

The basis of this report is the program system that is presently (1991) used at the National Land Survey of Sweden for geodetic control surveying, namely GPPS (version 91.07) from Ashtech Inc. The methods for analysis and the criteria given below are based on experiences with this system. A comparison has then been made between GPPS and the software of Trimble, TRIMVEC-PLUS, version C (Jivall 1991). The methods of analysis developed for GPPS turned out to work well also for TRIMVEC-PLUS. The criteria, however, had to be modified somewhat. Therefore, two sets of criteria will be presented here, one for GPPS and one for TRIMVEC-PLUS. The guide-lines, and to a certain extent also the criteria, should be applicable to other program systems as well, provided they are founded on similar fundamental principles.

4.2 Phase measurements and differences between these

The basic observation for accurate relative positioning is a phase observation of the carrier of the satellite signal. This can be expressed as a function of the position of the receiver, the position of the satellite, the number of complete cycles between the satellite and the receiver at the beginning of the phase measurement (cycle ambiguity), and the offsets of the satellite clock and the receiver clock. In a simplified manner the observation equation can be written

$$\Phi^s(t) = -(1/\lambda)r^s(t) + N^s + f(\Delta t + \Delta t^s)(t) \quad (11)$$

$$r^s = ((X^s - X_m)^2 + (Y^s - Y_m)^2 + (Z^s - Z_m)^2)^{1/2}$$

where	Φ	=	phase observation
	r	=	distance between satellite and receiver
	λ	=	wave-length
	N	=	cycle ambiguity
	f	=	frequency
	t	=	time
	s	=	satellite
	m	=	receiver

For each epoch one such observation is obtained for each combination of satellite and receiver. The position of the receiver and the ambiguity are independent of time whereas the clock offsets are time-dependent. The position of the satellite is normally considered as known; it is computed from the ephemeris.

A usual method for eliminating, or at least reducing, the effects of the clock offsets is to form differences between the phase observations. By *single differences* - differences between two phase observations for the same satellite but two different stations (figure 4) - the satellite clock offset is eliminated. Orbital errors and atmospheric effects are reduced as well.

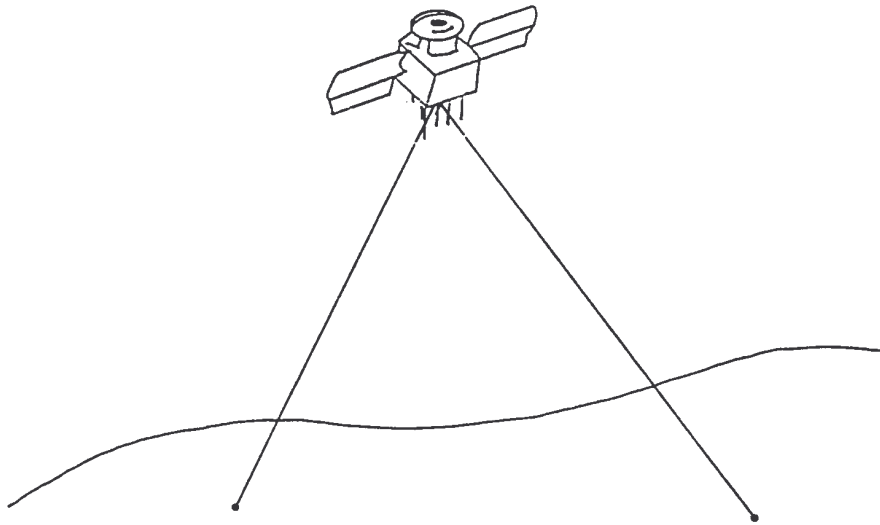


Figure 4. Single difference.

The difference between two single differences for the same stations but different satellites is called *double difference* (figure 5). Here also the receiver clock offsets are eliminated. In addition, double differences allow ambiguities to be distinguished from other parameters, making them suitable as observations in the final adjustment.

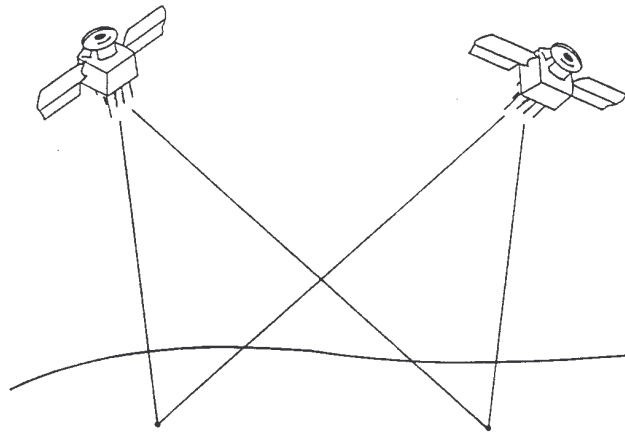


Figure 5. Double difference.

Triple differences are differences in time between two double differences. They are characterized by being independent of the ambiguities, since these are eliminated at the formation of the differences. Baseline computation with triple differences is less accurate than with double differences. Triple differences are useful for locating and repairing cycle slips, and for computing a baseline in a simple manner independently of ambiguities and cycle slips.

4.3 Ephemerides

When computing control surveys the positions of the satellites are almost without exception considered as known. Information on the satellite orbits is obtained either through Broadcast Ephemeris (predicted ephemeris) or through Precise Ephemeris (postcomputed ephemeris). Normally, Broadcast Ephemeris can be used. In the case of long baselines with high demands for accuracy, Precise Ephemeris should be used. The introduction of Selective Availability (SA) *may*, however, mean that Precise Ephemeris will be necessary also for ordinary control surveying.

4.4. Atmospheric corrections

Normally, a standard atmosphere with standard values should be used for the tropospheric corrections. Absolute tropospheric errors yield a scale error; however, deviations from the standard atmosphere result in maximum errors of only a few tenths of a ppm. Relative tropospheric errors influence, primarily, the height. If one aims at using own relative meteorological data, extremely accurate and careful observations are required to improve the computations.

Concerning the ionosphere, the effect of this can be eliminated with dual frequency measurements. However, often single frequency measurements are used. In that case, considerable parts of the systematic errors can be corrected by a ionosphere model. If no such model is used a scale error can be expected, ranging from -0.35 to -3.5 ppm.

4.5 Initial coordinates

Since the relationship between phase measurements (or differences thereof) and the receiver's coordinates is non-linear, approximate values are needed for the unknown coordinates. Bad approximate values will result in a bad solution, if the solution at all will converge. If the computation of baseline components, which is a relative computation, is performed in an erroneous absolute position, systematic errors will enter into the result. Height errors will result in a scale error, and horizontal errors in a rotation of the net (Beutler et al. 1987). The scale error is directly proportional to the height error; an absolute position 10 m too high gives rise to a scale 0.4 ppm too small. At the reduction to the ellipsoid the scale error is reinforced by 1.6 ppm per 10 m. Thus the total effect of a 10 m error in height becomes 2.0 ppm (cf. figure 6). The systematic errors due to an incorrect horizontal absolute position are considerably smaller (10 m yields $0.03'' = 0.009$ mgon). However, a large error (50 - 100 m) may give rise to convergence problems.

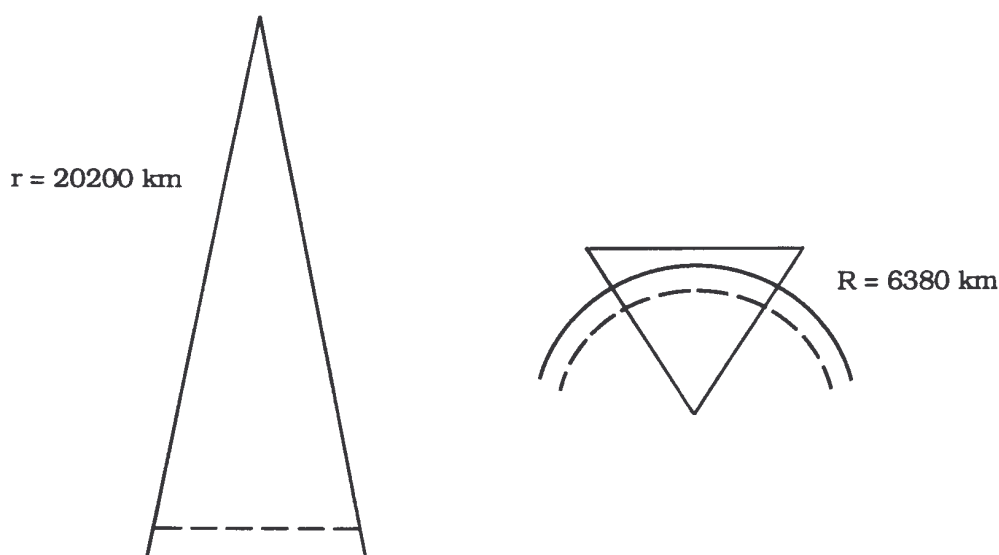


Figure 6. Scale errors due to erroneous initial height.

Generally speaking, absolute coordinates can be determined by absolute positioning with GPS or Transit Doppler, or by transformation from another coordinate system. Absolute positions from GPS with Broadcast Ephemeris are too inaccurate to be used as initial coordinates in control survey computations.

It is sufficient to have only one station with good initial coordinates. These are then propagated via the baseline computation to the other stations in the network. It is important that all calculations in a network are made in the same absolute position, otherwise unnecessary strain is introduced into the network.

The coordinates for the starting point are preferably determined by transformation from a national system (RT90 in Sweden) to WGS84 according to figure 1 and paragraph 3.1. Other methods with a corresponding accuracy might also be used.

$$(x, y, H)_{RT} \rightarrow (\varphi, \lambda, H)_{RT} \rightarrow (X, Y, Z)_{RT} \rightarrow (X, Y, Z)_{WGS}$$

It should be noted here that height errors (maximum c. 10 m) due to residuals in the 3-D Helmert transformation in paragraph 3.1 only effect the baseline components. The reduction to the ellipsoid becomes correct since it is made after the datum shift.

For all other stations the absolute position from GPS will suffice for initial coordinates. Better approximate coordinates are then usually computed from triple differences at a fairly early stage in the computational process.

In the computation systems there is often included a module that calculates absolute positions from pseudorange measurements. The position of the receiver and corrections to the clock of the receiver are determined through resection by distances in space. A corresponding computation is often already made in the receiver but a computation afterwards allows a satellite to be deleted or another ephemeris to be used. Besides, the clock corrections of the receiver can be improved, this being especially important for temperature stabilized oscillators.

4.6 Data filtering and adjustment of double differences, float and fix solutions

Before an adjustment is performed data should be cleaned from bad (deviating) observations, and cycle slips should be repaired. This is usually made with triple differences.

Subsequently an adjustment is carried out with double differences as observations, and coordinate differences and ambiguities as unknowns. For the adjustment one needs at least one hour of observations with a minimum of four simultaneous satellites (the time depends on the software, the baseline length and the satellite configuration). The main reason behind such a long time of observations is that the usual static method for determining cycle slips requires quite a large change of the satellite configuration during the measurement.

The specific adjustment part in the program system consists of two adjustments. First an adjustment is made where both coordinates and ambiguities are solved for. Since the ambiguities in such a solution become float numbers the solution is called *float solution*. In a good solution the ambiguities are close to integers. The ambiguities should by definition be integers; hence they are usually fixed at integers in a so-called *fix solution* in order to improve the result. The upper limit for fix solutions using L1 measurements is at a baseline length of 15 - 20 km. *In control surveying with such baselines only fix solutions are allowed.*

Before fixing the ambiguities a test is usually made to check these. All combinations of ambiguities, where one or several ambiguities have changed by a complete cycle, are tested through computing solutions with just a few observations. The ratio between the variances of the second best and the best solution is an important measure of the quality of the computation. A high ratio implies that the correct solution has been chosen.

The sizes of the RMS in the float and fix solutions, together with the difference between them, are also useful quality measures. Large values of the RMS, and a large increase in RMS between the float and fix solutions, indicate that incorrect ambiguities might have been chosen.

Paragraph 4.8 contains criteria for the above-mentioned quality measures. The main rule is that only solutions fulfilling the criteria should be treated further. It is, however, impossible to formulate criteria that reject all bad solutions and keep all good ones. Consequently, some solutions will not fulfill the criteria although they are good ones. Baselines being on the limit but not quite fulfilling all criteria may be kept, but in that case it is important to check them closely at later stages (controls within and between sessions, and in the network adjustment). Note that a prerequisite for these criteria to function properly is that a sufficient amount of data is at hand.

4.7 Baseline programs and multi-station programs

Some computation programs offer the possibility of adjusting several baselines in a simultaneous adjustment, so-called *multi-station programs* (e.g. PoPS from Wild-Magnavox and TRIMMBP from Trimble Navigation), whereas others can handle only one baseline at a time, so-called *baseline programs* (e.g. Ashtech's GPPS). Also in multi-station programs one can, of course, perform baseline adjustments.

Baseline programs have a simpler structure and require less memory. Bad baselines are easier to localize and to omit. In a recomputation only the baselines concerned have to be recomputed, unlike in a multi-station program where the whole session has to be recomputed. Since the baselines are treated separately in a baseline program one cannot find the correlations *between* the baselines, which is of importance in the subsequent network adjustment. *In order to, although in a somewhat primitive way, take care of the correlations between baselines, all combinations of baselines should be computed (i.e. also the trivial ones).* In case not all receivers have been in operation during precisely the same time span this method means, in addition, a better utilization of the data than in a multi-station processing.

In a multi-station program the correlations can be correctly modelled. The computation is faster since only $n-1$ baselines are to be computed, in comparison with $n(n-1)/2$ when computing all baseline combinations in a baseline program. However, the program requires more memory and the gross error detection is more complicated. Even in a multi-station program baselines are created or "defined", provided the adjustment is made with double differences. If so, it is recommendable to define the same baselines as those included in the planning of the network. In principle, a whole control network may be adjusted in one go but often limitations are put by the program (or the memory) in form of a maximum number of stations and/or sessions. Therefore, each session is usually adjusted separately to be, later on, combined with the other sessions in the network adjustment.

4.8 Criteria

Below, criteria are given for GPPS (Ashtech's software) as well as for TRIMVEC-PLUS (Trimble's software). For GPPS also normal values are given. The criteria are ordered approximately according to priority. The reader should note that a prerequisite for these criteria to function properly is a sufficient amount of data (at least one hour with at least four simultaneous satellites). If too many measurements are omitted the test values may very well be good but the solution, in spite of this, a bad one.

Explanations:

1. Ambiguities: The values for ambiguities refer to deviations from integers.
2. Test ratio: The test ratio is the quotient between the quadratic sums of the second best and the best solutions, respectively, where the difference between the solutions consists of one or a few ambiguities being fixed at other integers. A high ratio implies that the correct ambiguities have been chosen in the best solution.

3. RMS fix: RMS is, in principle, the same thing as the standard error of unit weight in the network adjustment. The criteria are given for fix solutions since these are the ones to be used in control surveying. Also the RMS difference between the float and fix solutions should be checked; a large increase in RMS indicates that incorrect ambiguities might have been used.
4. Diff float-fix: The values refer to maximum differences in each component (N, E, U or X, Y, Z) between the float and fix solutions.

PARAMETER	CRITERIA (normal values)	
	0 - 10 km	10 - 30 km
Baseline length:	0 - 10 km	10 - 30 km
1.Ambiguities	< 0.20 cycles (< 0.10 cycles)	< 0.25 cycles (< 0.15 cycles)
2.Test ratio	> 3 (4 - 100)	> 2 (3 - 50)
3.RMS-fix	<0.08 cycles=15 mm (0.02-0.06 cycles) (= 4-11 mm)	<0.10 cycles=19 mm (0.04-0.08 cycles) (= 8-15 mm)
4.Diff float-fix	< 5 cm (0-4 cm)	< 7 cm (0-6 cm)

Table 1. Criteria for the adjustment of GPS observations in GPPS.

PARAMETER	CRITERIA	
	0 - 10 km	10 - 30 km
Baseline length:	0 - 10 km	10 - 30 km
1.Ambiguities	< 0.25 cycles	< 0.30 cycles
2.Test ratio	> 3	> 2
3.RMS-fix	<0.06 cycles=11 mm	<0.07 cycles=13 mm
4.Diff float-fix	< 5 cm	< 7 cm

Table 2. Criteria for the adjustment of GPS observations in TRIMVEC-PLUS.

4.9 Guide-lines in brief

- For the computation of a session or a baseline one needs at least one hour of observations with at least four simultaneous satellites (the time depends on the software, the baseline length and the satellite configuration).
- Normally, use Broadcast Ephemeris. When computing long baselines with high demands for accuracy, Precise Ephemeris should be used. The introduction of SA (Selective Availability) might raise new demands.
- The adjustment of GPS observations must be made in a correct absolute position. Initial coordinates for a starting point are obtained through transformation from a national system to WGS84 (see chapter 3) or with some other method yielding at least the same accuracy. Even more important than the absolute position itself is that all computations in a network are performed in the same absolute position.
- Normally, use a standard atmosphere for troposphpherical corrections.
- For control surveyys only fix solutions are allowed.
- Only solutions fulfilling the criteria in paragraph 4.8 should be treated in the network adjustment.
- When using baseline programs all baselines (also the trivial ones) must be computed.

5. CONTROLS WITHIN AND BETWEEN SESSIONS

5.1 Baseline programs

In order to check that the baselines within each session agree, one network adjustment per session is made. The residuals in such an adjustment should be small (a few mm) as the baselines originate from common observations (trivial baselines). Errors in the computation of a single baseline can be detected in this way.

The sessions are then checked against each other by comparing double-measured baselines. Of course one may compute loop misclosures instead but this is often more time-consuming. Centering errors usually can be detected in this step.

5.2 Multi-station programs

In a multi-station program the control within the session has already been made as the whole session is adjusted simultaneously.

Controls between sessions can be made in the same way as for baseline programs provided one utilizes the trivial baselines as well. Alternatively, the check is made by computing misclosures in loops (squares).

5.3 Error limits

Below, error limits are given that are found empirically from double-measured baselines between 100 m and 20 km. Standard errors for each component have been estimated in a local system (northing, easting, up); from these the error limits have then been derived. For the error limits we employ the approach with a *warning limit* at two sigma (2σ) and a *rejection limit* at three sigma (3σ) for one-dimensional quantities, and a corresponding approach in two and three dimensions (σ = standard deviation). Baselines having differences larger than the warning limit shall be checked (input and adjustment of GPS observations) and, if necessary, corrected. If no error is found the baseline is kept provided the differences are below the rejection limit, otherwise it is omitted.

When deriving error limits for loop misclosures we have taken into consideration the fact that the baselines in a loop are normally of unequal lengths. Thus the error limits for loop misclosures are not to be applied for double-measured baselines, although these can be looked upon as loops consisting of two vectors.

5.3.1 Error limits for double-measured baselines

$$D = a + bl \tag{12}$$

D = limit for difference in mm between double-measured baselines

a = constant in mm

b = constant in mm/km

l = length in km

	Warning limit		Rejection limit	
	a	b	a	b
Northing	10	2	15	3
Easting	6	2	9	3
Up	20	3.4	30	5.1
Horizontal	11	2.6	15	3.6
3-D	23	4.3	30	5.6

Table 3. Constants of error limits for double-measured baselines.

5.3.2 Error limits for loop misclosures

$$D = (an + bL)/(n)^{1/2} \quad (13)$$

D = limit for loop misclosure in mm

a = constant in mm

b = constant in mm/km

n = number of baselines in loop

L = total length in km

	Warning limit		Rejection limit	
	a	b	a	b
Northing	8	1.6	11	2.4
Easting	5	1.6	7	2.4
Up	15	2.7	22	4.1
Horizontal	8	2.1	11	2.9
3-D	17	3.4	22	4.6

Table 4. Constants of error limits for loop misclosures.

6. NETWORK ADJUSTMENT

6.1 General aspects

The baseline components (from baseline or multi-station programs) or the coordinate sets for each session/day (from multi-station programs) must be adjusted to a unified network for the whole project. To start with, a free adjustment is always made to make it possible to check the accuracies of the GPS stations and the connection stations separately. Subsequently, an adjustment might be performed with the connection stations more or less fixed, a subject to which we will return in the next chapter.

At the National Land Survey of Sweden we use at present (1991) the adjustment program GeoLab, which performs a simultaneous adjustment in three dimensions in geocentric cartesian coordinates. In some other adjustment programs the baselines are first transformed into length, azimuth and zenith distance, after which either a common adjustment is made or an adjustment separated into a horizontal and a vertical part. The latter type of adjustment programs will not be treated here. In the case of horizontally and vertically separate adjustments there should, however, exist a coupling between the two adjustments, making sure that the baselines are treated in the same way in both cases.

6.2 Correlations and weighting

Since the same observation data have been used in computing several baselines the baselines become correlated with each other. Also within the baselines, between the various components, there are correlations. From a multi-station program normally correlations within as well as between the baselines are delivered, whereas a baseline program only yields correlations within the baselines.

In order to, although in a somewhat primitive way, take care of the correlations between the baselines from a baseline program, *all combinations of baselines (i.e. also the trivial ones) must be included in the network adjustment.*

The weighting can be made either according to standard weighting or according to the internal standard errors from the adjustment of GPS observations. In the latter case the standard errors should be rescaled to agree, as much as possible, with the standard weighting given below.

Based on the same material as was used for determining error limits for double-measured baselines, the following standard weighting (a priori standard errors) has been determined:

$$\begin{aligned}\sigma_N &= 5 \text{ mm} + 0.7 \text{ ppm} \\ \sigma_E &= 5 \text{ mm} + 0.7 \text{ ppm} \\ \sigma_U &= 8 \text{ mm} + 1.2 \text{ ppm}\end{aligned}$$

In some network adjustment programs (e.g. GeoLab) the weighting must be given in geocentric cartesian coordinates. To get the weights in this form one can apply the law of propagation of standard errors on the formulae (10) in chapter 3. Alternatively formulae common for the whole country may be used; in the case of Sweden they are derived for $\phi = 62^\circ$, $\lambda = 16^\circ$:

$$\begin{aligned}\sigma_x &= 6 \text{ mm} + 0.8 \text{ ppm} \\ \sigma_y &= 5 \text{ mm} + 0.7 \text{ ppm} \\ \sigma_z &= 7 \text{ mm} + 1.1 \text{ ppm}\end{aligned}$$

If standard weighting is adopted the correlations from the adjustment of GPS observations should not be included in the network adjustment.

6.3 Analysis

The network adjustment can be analysed with the help of the following quantities:

- standard errors of unit weight
- residuals/standardized residuals
- station standard errors.

The *standard error of unit weight* should be close to 1 if the weighting is correct. The neglect of correlations between baselines yields, however, a slightly lower standard error. If all combinations of baselines are adjusted, without regard to the correlations, one can expect a standard error of unit weight of about 0.9.

The *residuals* give in absolute figures (even if the scale of the weighting would be wrong) estimates of the discrepancies in the network. The residuals are, therefore, one of the most important measures of accuracy.

Standardized residuals are residuals divided by their respective standard errors. Large standardized residuals may indicate gross errors. On the analogy of the error limits for double-measured baselines we put the *warning and rejection limits for the standardized residuals to two and three, respectively*; i.e. the limits (tolerances) for the residuals are 2σ and 3σ . Here we presuppose that the standardized residuals are calculated with the *a priori standard errors* of the residuals, which is the case in, among other programs, GeoLab (up to version 1.91). If the *a posteriori* standard error is used the error limits have to be divided by the standard error of unit weight.

Baselines with standardized residuals above the warning limit shall be investigated and, if possible, corrected. If no error is found the baseline is kept but down-weighted, provided the standardized residual is below the rejection limit; otherwise it is omitted.

If the origin of the large residuals is a height error, and the heights will not be used, one may keep the baseline unchanged in case the height is less than 2 dm. This should then be commented upon in the report. If it is possible one should, however, correct the error. Height errors can be detected by transforming the residuals with formula (9) in paragraph 3.2.

Changes like down-weighting and rejection shall be made for one unit at a time, namely the one with the largest standardized residual.

A unit could here be a baseline, a station or a session depending on what type of error that has caused the high residuals. The errors can be divided into the following groups:

- Baseline-dependent errors, e.g. errors arisen in the baseline computation.
- Station-dependent errors, e.g. centering errors, receiver-dependent errors or errors due to bad receiving conditions (obstructed sight to the satellites etc.).
- Session-dependent errors, e.g. errors arisen in a multi-station adjustment, errors due to atmospherical disturbances or bad ephemerides.

In order to keep control of the weaknesses, if any, of the network, rejected and down-weighted baselines should be marked in a network sketch.

The down-weighting is made according to the following formula.

$$\sigma'_m = v/r \quad (14)$$

$$r = (\sigma_v/\sigma_m)^2 \quad (15)$$

$$\sigma'_m = v(\sigma_m/\sigma_v)^2 \quad (16)$$

where

- v = residual
- σ_m = a priori standard error of a measurement
- σ'_m = new a priori standard error of a measurement
- σ_v = a priori standard error of a residual
- r = redundancy number

By down-weighting, the whole baseline (all three components) must be weighted down with the same factor.

Note that the down-weighting has to be made for one deviator at a time, the one with the largest standardized residual.

Relative station standard errors may give information about low accuracy stations in the network. To make the interpretation correct the internal weighting must be correct. The station standard errors do not tell anything, however, about the controllability of the network. Therefore, a graphic presentation of the network with down-weighted baselines marked is an important complement.

If all combinations of baselines have been included in the network adjustment the station standard errors must be rescaled to become more realistic. These then have to be multiplied by the factor

$$f = (n_{tot}/n_{1-t})^{1/2} \quad (17)$$

where

- n_{tot} = total number of baselines
- n_{1-t} = number of non-trivial baselines

7. CONNECTION TO A SUPREME NETWORK

7.1 Two different methods

For the final connection to a supreme network one can imagine two principally different methods, namely a Helmert transformation of a freely adjusted network or a network adjustment where the connection stations are held fixed. The latter method first requires a check through a Helmert transformation (or through an equivalent procedure).

At the National Land Survey of Sweden almost exclusively the Helmert transformation method has been applied for GPS networks. For conventional networks (distances and angles) the connection stations are usually held fixed if they are of at least the same accuracy as the new network.

Since we have such limited experience from GPS network adjustment with fixed stations we give no recommendations on what method to use in various cases. A brief discussion on the advantages and disadvantages of the two methods may, however, be appropriate.

The advantage of the Helmert transformation of a free network is the robustness and simplicity of the method. The internal accuracy in the new network is kept. When adjusting with fixed stations, distortions in the new network may sometimes occur. Errors in the connection stations may give rise to even larger errors in the new network in the case of certain weak network configurations. If the connection stations are less accurate than the new network the method with fixed stations will of course make the internal accuracy of the new network lower.

The great advantage with network adjustment with fixed stations is the continuous transition to a supreme network. When using the Helmert transformation there appears a gap, corresponding to the residuals of the transformation, between the connection stations and the new stations. To overcome this problem with the Helmert transformation one might introduce some kind of compensation for the residuals.

7.2 Helmert transformation of a freely adjusted network

The coordinates in an approximate WGS84 from a free network adjustment are transformed with the help of the datum shift in paragraph 3.1 to approximate coordinates in a national system (in Sweden RT90; see further figure 1 in chapter 2).

$$(X,Y,Z)_{\text{WGS}} \rightarrow (X,Y,Z)_{\text{RT}} \rightarrow (\varphi,\lambda,H)_{\text{RT}} \rightarrow (x,y)_{\text{RT}}$$

The approximate national coordinates are then transformed to the supreme network with a plane Helmert transformation (translation, rotation and scale). The two coordinate sets involved in the transformation of course have to refer to the same projection system.

The standard error of unit weight, the scale, the rotation and the transformation residuals are measures of how well the two networks agree. They allow one to detect, among other things, if any of the connection stations is bad. Statistical tests of residuals and standard errors, estimated errors etc. are very useful in the error detection.

The following values are normal for a Helmert transformation (of Ashtech measurements processed with GPPS) to 4-5 stations in the Swedish national triangulation network (RT90):

standard error	
of unit weight:	10-20 mm/coordinate
scale:	1-3 ppm (cf. paragraph 4.4)
rotation:	0.0-0.4 mgon

7.3 Network adjustment with fixed stations

First a Helmert transformation is made according to paragraph 7.2 with the purpose of checking the connection stations. All connection stations are transformed to WGS84 according to formula (4) in chapter 3. Here it is important to use the true inverse of the transformation WGS \rightarrow RT (and not only change signs of translations and rotations) to get exactly the same coordinates when returning to RT after the network adjustment.

In the network adjustment the latitudes and longitudes of the connection stations are fixed at the obtained WGS84 values. Only one station is fixed vertically, thereby avoiding the less accurate heights to influence the horizontal accuracy. In GeoLab there is no possibility to fix only the latitudes and longitudes; instead you give them a high weight, i.e. a low a priori standard error.

In case the Helmert transformation has yielded a significant scale and rotation, these parameters may be solved for in the network adjustment. After the network adjustment the coordinates of the new stations are transformed to horizontal coordinates in the same way as in paragraph 7.2. However, no Helmert transformation is made.

8. CONCLUDING REMARKS

The guide-lines, criteria and error limits given in this report are derived from experiences of the measurements of the National Land Survey of Sweden during the latest years. They are, therefore, primarily adapted to the system in use at the National Land Survey, i.e. network design according to Lithén & Persson (1991), adjustment of GPS observations with Ashtech's baseline program GPPS and network adjustment with the program GeoLab. Concerning the criteria for the adjustment of GPS observations these have been adapted to the program TRIMVEC-PLUS as well. With slight modifications the guide-lines and criteria should be applicable also to other systems based on the same fundamental principles. The error limits are more general.

Research and development concerning methods of computation are intense for the moment and for that reason the guide-lines in this report should not be looked upon as valid for ever. Especially in the chapter on adjustment of GPS observations one may anticipate changes. Faster methods for solving ambiguities are being developed at present. Our own increased experiences will probably also lead to changes.

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APPENDIX A: Transformation between X, Y, Z and φ, λ, h

The relation between the geocentric cartesian coordinates X, Y, Z and the geodetic coordinates φ, λ, h can be written:

$$X = (N + h) \cos \varphi \cos \lambda \quad (\text{A.1})$$

$$Y = (N + h) \cos \varphi \sin \lambda \quad (\text{A.2})$$

$$Z = (N(1 - e^2) + h) \sin \varphi \quad (\text{A.3})$$

with

$$N = \frac{a}{(1 - e^2 \sin^2 \varphi)^{1/2}} \quad (\text{A.4})$$

$$e^2 = f(2 - f) \quad (\text{A.5})$$

where

- X,Y,Z = geocentric cartesian coordinates
- φ, λ, h = geodetic coordinates
- N = normal radius of curvature
- e = first excentricity
- f = flattening
- a = semi-major axis

The inverse relation can be found in the following way. The longitude λ is easily calculated as

$$\tan \lambda = Y/X \quad (\text{A.6})$$

The latitude φ is more complicated to calculate. Several approaches exist. One of them (Leick 1990) is the iterative formula

$$\tan \varphi_{i+1} = \frac{Z}{(X^2 + Y^2)^{1/2}} \left[1 + \frac{e^2 N \sin \varphi_i}{Z} \right] \quad (\text{A.7})$$

The starting value of φ_i on the right-hand side is, thereby,

$$\tan \varphi_0 = \frac{Z}{(1 - e^2)(X^2 + Y^2)^{1/2}} \quad (\text{A.8})$$

Formula (A.8) is derived by exchanging Z within the brackets in formula (A.7) for formula (A.3) with $h = 0$.

The height h is obtained through

$$h = \frac{(X^2 + Y^2)^{1/2}}{\cos \varphi} - N \quad (\text{A.9})$$

As has been mentioned above there are several ways to carry out the transformation X, Y, Z \rightarrow φ, λ, h ; also closed formulae exist.

APPENDIX B: Transverse Mercator projection

This projection is also known as Gauss' conformal projection or the Gauss-Krüger projection.

The formulae cannot be written in closed form but only as series expansions. The formulae given here are taken from Ussisoo (1977) and Krüger (1912); they have millimeter accuracy for $|y - y_0| < 700$ km.

The projection formulae make use of the isometric latitude φ^* . Below, the relations between the isometric latitude φ^* and the geodetic latitude φ are given.

$$\varphi^* - \varphi = A \sin 2\varphi + B \sin 4\varphi + C \sin 6\varphi \quad (\text{B.1})$$

$$\varphi - \varphi^* = A^* \sin 2\varphi^* + B^* \sin 4\varphi^* + C^* \sin 6\varphi^* \quad (\text{B.2})$$

with

$$\begin{aligned} A &= -2n + (2/3)n^2 + (4/3)n^3 + \dots \\ B &= (5/3)n^2 - (16/15)n^3 + \dots \\ C &= -(26/15)n^3 + \dots \\ A^* &= 2n - (2/3)n^2 - 2n^3 + \dots \\ B^* &= (7/3)n^2 - (8/5)n^3 + \dots \\ C^* &= (56/15)n^3 + \dots \end{aligned} \quad (\text{B.3})$$

$$n = f/(2 - f)$$

where φ = geodetic latitude
 φ^* = isometric latitude
 f = flattening

Note that the angle $\varphi - \varphi^*$ in the formulae (B.1) and (B.2) is expressed in radians.

$\varphi, \lambda \rightarrow x, y$

$\varphi \rightarrow \varphi^*$ according to (B.1)

$$\tan \xi' = \tan \varphi^* / \cos (\lambda - \lambda_0) \quad (\text{B.4})$$

$$\tanh \eta' = \cos \varphi^* \sin (\lambda - \lambda_0) \quad (\text{B.5})$$

$$x = k_0 \hat{a} (\xi' + \beta_1 \sin 2\xi' \cosh 2\eta' + \beta_2 \sin 4\xi' \cosh 4\eta' + \beta_3 \sin 6\xi' \cosh 6\eta') \quad (\text{B.6})$$

$$y = y_0 + k_0 \hat{a} (\eta' + \beta_1 \cos 2\xi' \sinh 2\eta' + \beta_2 \cos 4\xi' \sinh 4\eta' + \beta_3 \cos 6\xi' \sinh 6\eta') \quad (\text{B.7})$$

$x, y \rightarrow \varphi, \lambda$

$$\xi = x / (k_0 \hat{a}) \quad (\text{B.8})$$

$$\eta = (y - y_0) / (k_0 \hat{a}) \quad (\text{B.9})$$

$$\xi' = \xi - \delta_1 \sin 2\xi \cosh 2\eta - \delta_2 \sin 4\xi \cosh 4\eta - \delta_3 \sin 6\xi \cosh 6\eta \quad (\text{B.10})$$

$$\eta' = \eta - \delta_1 \cos 2\xi \sinh 2\eta - \delta_2 \cos 4\xi \sinh 4\eta - \delta_3 \cos 6\xi \sinh 6\eta \quad (\text{B.11})$$

$$\sin \varphi^* = \sin \xi' / \cosh \eta' \quad (\text{B.12})$$

$$\tan (\lambda - \lambda_0) = \sin \eta' / \cos \xi' \quad (\text{B.13})$$

$\varphi^* \rightarrow \varphi$ according to (B.2)

Here x, y = grid coordinates
 φ, λ = geodetic latitude and longitude
 λ_0 = central meridian
 y_0 = false easting
 k_0 = scale factor along the central meridian

In the formulae above we further have:

$$\hat{a} = 2Q/\pi = \frac{a}{(1+n)} (1 + (1/4)n^2 + (1/64)n^4 + \dots) \quad (\text{B.14})$$

$$\beta_1 = (1/2)n - (2/3)n^2 + (5/16)n^3 + \dots$$

$$\beta_2 = (13/48)n^2 - (3/5)n^3 + \dots$$

$$\beta_3 = (61/240)n^3 + \dots$$

$$\delta_1 = (1/2)n - (2/3)n^2 + (37/96)n^3 + \dots$$

$$\delta_2 = (1/48)n^2 + (1/15)n^3 + \dots$$

$$\delta_3 = (17/480)n^3 + \dots$$

where Q = meridian quadrant
 a = semi-major axis
 n is obtained from (B.3)