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THE NEED FOR ASTRONOMICAL OBSERVATIONS IN THE DENSIFICATION OF SATELLITE POSITIONING NETWORKS

A THEORETICAL STUDY

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THE NEED FOR ASTRONOMICAL OBSERVATIONS IN THE DENSIFICATION OF SATELLITE POSITIONING NETWORKS. A THEORETICAL STUDY

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ABSTRACT

Today satellite positioning (Doppler or GPS) may more or less replace classical, optical astronomical observations. The problem whether there is any need for Laplace azimuths in the densification of a framework of satellite fixes by EDM traverses is studied numerically. The conclusion is that the astronomical observations can normally be left out without any problem. In the future it is expected that geodetic astronomy will find its main utility in controlling the absolute orientations of satellite positioning systems.

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1. Introduction

Traditionally astronomical observations have played an essential role for the orientation of horisontal geodetic networks. This is achieved by the determination of the Laplace azimuth from one triangulation point to another. These azimuths serve to orient the network in an absolute astronomical coordinate system, i.e. with respect to a system of fixed stars. Moreover the astronomical observations of latitude and longitude have fruitfully promoted the knowledge of the deflections of the vertical and the interpolation of geoidal heights, quantities needed for the reduction of most terrestrial observations to the reference ellipsoid, where, usually the adjustment of a two-dimensional horisontal network takes place.

Today the geoid and the deflections of the vertical are more easily determined by terrestrial gravimetric observations in combination with satellite harmonic coefficients of the geopotential and satellite Doppler derived geoidal heights. Furthermore the satellite positioning methods (the currently available Transit system and the forthcoming Navstar system) offer excellent opportunities for the establishment of new national frameworks in developing countries solely based on satellite fixes. The satellite Doppler fixes so derived may be determined to an accuracy of ±1 meter in a global ("geocentric") coordinate system. Hence the relative accuracy of one part in 200 000 of fixes 200 km apart is quite realistic and may directly serve as the framework for further breakdowns by terrestrial observations. This is equivalent to 1" in the mutual azimuth, which is a number of present geodetic standard for this distance (see Bomford, 1980, p. 13-14). Consequently there is hardly any need for further Laplace azimuth controls. The Doppler fixes 200 km or more apart also serve well for the study of scale and orientation in already well established networks in developed countries. Subsequently in many countries (such as Sweden) the satellite Doppler observations have (at least for the time being) taken over all the resources for current progress in Laplace azimuth determination.

It should be noted that a satellite positioning system provides a world-wide unique reference frame for national or international networks. The fact that the orientation of such a system may differ somewhat from an astronomically defined system should not be a great problem. A possible transformation of the satellite derived framework to an astronomical framework is best solved in an international cooperation and not by each country alone.

Let us now assume that a national framework of satellite fixes approximately 200 km apart has been established. For the densification of the framework the satellite fixes might be kept fixed (infinite weight), or they might be included in an adjustment of the densified network. The densification is most practically achieved by EDM traversing. As there is no visual connection between the Doppler or GPS fixes the question arises whether astronomical azimuths are needed at the fixes for the densification. Another question is whether the traverses should be stabilized by observed azimuths on the middle between the fixes or whether Doppler or GPS observations should be preferred. These problems will be studied in the rest of this paper. In section 2 we derive the necessary formulas for studying the error propagations in an EDM traverse. In section 3 we give numerical results, and, finally, in section 4 we present our conclusions.

2. Derivation of formulas

Consider the traverse given in Figure 1. The following equations are easily obtained from the traverse

$$x_n = x_1 + \sum_{i=1}^{n-1} d_i \cos \varphi_i \tag{1}$$

$$x_n = x_N - \sum_{i=n}^{N-1} d_i \cos \varphi_i \tag{2}$$

$$\sum_{i=1}^{N} \beta_i \pm N\pi + \varphi_0 - \varphi_N = 0 \tag{3}$$

$$y_n = y_1 + \sum_{i=1}^{n-1} d_i \sin \varphi_i \tag{4}$$

$$y_n = y_N + \sum_{i=n}^{N-1} d_i \sin \varphi_i \tag{5}$$

$$x_n = x_m + \begin{cases} \sum_{i=m}^{n-1} d_i \cos \varphi_i , & m < n \\ 0 & m = n \\ -\sum_{i=n}^{m-1} d_i \cos \varphi_i & m > n \end{cases} \tag{6}$$

$$y_n = y_m + \begin{cases} \sum_{i=n}^{m-1} d_i \sin \varphi_i & m < n \\ 0 & m = n \\ -\sum_{i=m}^{n-1} d_i \sin \varphi_i & m > n \end{cases} \tag{7}$$

$$\sum_{i=1}^{m} \beta_i \pm m\pi + \varphi_0 - \varphi_m = 0 \tag{8}$$

where

$$\varphi_i = \varphi_0 + \sum_{k=1}^{i} \beta_k \pm i\pi \tag{9}$$

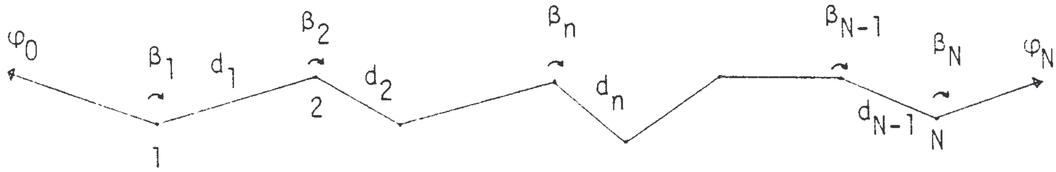

Figure 1. Traverse

Formulas (1), (2), (4) and (5) represent the usual determination of x_n and y_n from the end point coordinates. Formula (3) is the condition for directions in a closed traverse and formula (8) is an extra condition provided by a known azimuth φ_m. Finally, the equations (6) and (7) are valid for known coordinates at P_m.

We now assume that the traverse is straight and approximately directed along the y-axes. Then we have

$$\cos \varphi_i \approx 0 \quad \text{and} \quad \sin \varphi_i \approx 1$$

Furthermore we assume that the traverse leg is constant, i.e. $d_i = d$ for all i. Under these assumptions the equations (1)-(8) correspond to the following condition equations with unknown corrections dx_n and dy_n to the approximate coordinates at P_n:

$$\varepsilon_{x_1} - d \sum_{i=1}^{n-1} \varepsilon_{\varphi_i} + dx_n = w_1 \tag{1'}$$

$$\varepsilon_{x_N} + d \sum_{i=n}^{N-1} \varepsilon_{\varphi_i} + dx_n = w_2 \tag{2'}$$

$$\sum_{i=1}^{N} \varepsilon_i + \varepsilon_{\varphi_0} - \varepsilon_{\varphi_N} = w_3 \tag{3'}$$

$$\varepsilon_{y_1} + \sum_{i=1}^{n-1} e_i + dy_n = w_4 \tag{4'}$$

$$\varepsilon_{y_N} - \sum_{i=n}^{N-1} e_i + dy_n = w_5 \tag{5'}$$

$$\varepsilon_{x_m} + \begin{Bmatrix} -d \sum_{i=m}^{n-1} \varepsilon_{\varphi_i} \\ 0 \\ d \sum_{i=n}^{m-1} \varepsilon_{\varphi_i} \end{Bmatrix} + dx_n = w_6 \qquad \begin{matrix} m < n \\ m = n \\ m > n \end{matrix} \tag{6'}$$

$$\varepsilon_{y_m} + \begin{Bmatrix} \sum_{i=m}^{n-1} e_i \\ 0 \\ - \sum_{i=n}^{m-1} e_i \end{Bmatrix} + dy_n = w_7 \qquad \begin{matrix} m < n \\ m = n \\ m > n \end{matrix} \tag{7'}$$

$$\sum_{i=1}^{m} \varepsilon_i + \varepsilon_{\varphi_0} - \varepsilon_{\varphi_m} = w_8 \tag{8'}$$

where ε_{x_i}, ε_{y_i} are the errors of approximate coordinates, ε_{φ_i} the errors of approximate directions, e_i the errors of approximate distances and w_i misclosures. Formula (9) yields the following relation between ε_{φ_i} and the errors ε_i of directly observed angles (β_i):

$$\varepsilon_{\varphi_i} = \varepsilon_{\varphi_0} + \sum_{k=1}^{i} \varepsilon_k \quad \text{or} \quad \varepsilon_{\varphi_i} = \varepsilon_{\varphi_N} - \sum_{k=i+1}^{N} \varepsilon_k \tag{9'}$$

Hence

$$\sum_{i=1}^{n-1} \varepsilon_{\varphi_i} = (n-1)\,\varepsilon_{\varphi_0} + \sum_{i=1}^{n-1} (n-i)\,\varepsilon_i \tag{10}$$

and

$$\sum_{i=n}^{N-1} \varepsilon_{\varphi_i} = (N-n)\,\varepsilon_{\varphi_N} - \sum_{i=n+1}^{N} (i-n)\,\varepsilon_i \tag{11}$$

Also

$$\sum_{i=m}^{n-1} \varepsilon_{\varphi_i} = (n-m)\,\{\varepsilon_{\varphi_0} + \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_m\} + \begin{cases} \sum_{i=m+1}^{n-1} (n-i)\,\varepsilon_i & m < n-1 \\ 0 & m = n-1 \end{cases} \tag{12}$$

and

$$\sum_{i=n}^{m-1} \varepsilon_{\varphi_i} = (m-n)\,\{\varepsilon_{\varphi_0} + \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_n\} + \begin{cases} \sum_{i=n+1}^{m-1} (m-i)\,\varepsilon_i & m > n+1 \\ 0 & m = n+1 \end{cases} \tag{13}$$

It can be seen from formulas (1')-(8') that the error in x_n stems from the propagated errors in x_1, x_m, x_N, φ_0, φ_m, φ_N and β_i, while error in y_n stems from errors in y_1, y_m, y_N and d_i. Assuming that there is no correlation between those two types of error sources the errors for x_n and y_n can be derived separately. All observation errors will be treated as random.

Following Bjerhammar (1973, ch. 20) the covariance matrix Q_{XX} of the adjusted unknowns X in a system of condition equations with unknowns

$$A X + B \varepsilon = W \tag{14}$$

where A and B are known coefficient matrices, ε vector of random observation errors and W is the vector of misclosures, is given by

$$Q_{XX} = \sigma^2 (A^T C^{-1} A)^{-1} \tag{15}$$

where

$$C = B Q B^T \tag{16}$$

Here Q is the covariance matrix of the observations and σ^2 is the variance of unit weight. Below σ^2 is set to unity.

2.1 The variance of y_n

y_n is determined by formulas (4'), (5') and (7'). Considering (14)-(16) with $A^T = (1,1,1)$ we thus obtain the elements of C

$$C_{11} = \sigma^2_{y_1} + (n-1)\, \sigma^2_d$$

$$C_{12} = C_{21} = 0$$

$$C_{13} = C_{31} = \begin{cases} 0 & m \geq n \\ (n-m)\, \sigma^2_d & m < n \end{cases}$$

$$C_{22} = \sigma^2_{y_N} + (N-n)\, \sigma^2_d$$

$$C_{23} = C_{32} = \begin{cases} 0 & m \leq n \\ (m-n)\, \sigma^2_d & m > n \end{cases}$$

and

$$C_{33} = \sigma^2_{y_m} + |n-m| \; \sigma^2_d$$

where $\sigma^2_{y_i}$, $\sigma^2_{x_i}$, etc. are obvious variances corresponding to random errors ε_{y_i}, ε_{x_i}, etc.

2.2 The variance of x_n

x_n is determined by formulas (1)-(3), (6) and (8). Considering (15) we thus obtain the following variance of the solution for x_n:

$$\sigma^2_{x_n} = (A^T \; C^{-1} \; A)^{-1}$$

where

$$A^T = (1,1,0,1,0)$$

$$C_{11} = \sigma^2_{x_1} + d^2(n-1) \; \sigma^2_{\varphi_0} + d^2 \; \sigma^2_\beta \; f(n-1)$$

$$C_{12} = C_{21} = 0$$

$$C_{13} = C_{31} = -d^2 \; (n-1) \; \sigma^2_{\varphi_0} - d^2 \; \sigma^2_\beta \; g(n-1)$$

$$C_{14} = C_{41} = d^2 \left[(n-m)(n-1) \; \sigma^2_{\varphi_0} + (n-m) \; \sigma^2_\beta \begin{Bmatrix} g(n-1) - g(n-m-1) \\ g(n-1) \end{Bmatrix} + \right.$$

$$\left. + \begin{Bmatrix} \sigma^2_\beta \; f(n-m-1) \\ 0 \end{Bmatrix} \right] \quad \begin{matrix} m < n \\ m \geq n \end{matrix}$$

$$C_{15} = C_{51} = -d^2\sigma^2_{\varphi_0}\ (n-1) - d^2\sigma^2_\beta\ \ g(n-1) + \begin{cases} d^2\sigma^2_\beta\ g(n-m-1) & m < n \\ 0 & m \geq n \end{cases}$$

$$C_{22} = \sigma^2_{x_N} + d^2\sigma^2_{\varphi_N}\ (N-n)^2 + d^2\sigma^2_\beta\ f(N-n)$$

$$C_{23} = C_{32} = d^2\sigma^2_{\varphi_N}\ \ (N-n) - d^2\sigma^2_\beta\ g(N-n)$$

$$C_{24} = C_{42} = \begin{cases} 0 & m \leq n \\ -d^2\sigma^2_\beta\ k(n,m) & m > n \end{cases}$$

$$C_{25} = C_{52} = \begin{cases} 0 & m \leq n \\ -d^2\sigma^2_\beta\ g(m-n) & m > n \end{cases}$$

$$C_{33} = d^2\ \{\sigma^2_{\varphi_0} + \sigma^2_{\varphi_N} + N\ \sigma^2_\beta\}$$

$$C_{34} = C_{43} = -\ (n-m)\ d^2\sigma^2_{\varphi_0} - d^2\sigma^2_\beta \begin{cases} (n-m)\ m + g(n-m-1) & m < n \\ 0 & m = n \\ (n-m)\ n + g(m-n-1) & m > n \end{cases}$$

$$C_{35} = C_{53} = d^2(\sigma^2_{\varphi_0} + m\ \sigma^2_\beta)$$

$$C_{44} = \sigma^2_{x_m} + (n-m)^2\ d^2\sigma^2_{\varphi_0} + d^2\sigma^2_\beta \begin{cases} (n-m)^2\ m + f(n-m-1) & m < n \\ 0 & m = n \\ (n-m)^2\ n + f(m-n-1) & m > n \end{cases}$$

$$C_{45} = C_{54} = -\ (n-m)\ d^2\sigma^2_{\varphi_0} + d^2\sigma^2_\beta \begin{cases} m(m-n) & m \leq n \\ n(m-n) + g(m-n-1) & m < n \end{cases}$$

$$C_{55} = d^2(\sigma^2_{\varphi_0} + \sigma^2_{\varphi_m} + m\ \sigma^2_\beta)$$

Furthermore

$$g(\ell) = 1 + 2 + \ldots + \ell = \ell(\ell+1)/2$$

$$f(\ell) = 1^2 + 2^2 + \ldots + \ell^2 = (2\ell^3 + 3\ell^2 + \ell)/6$$

and

$$k(n,m) = \begin{cases} 0 & m = n+1 \\ \sum_{i=n+1}^{m-1} (m-i)(i-n) & m > n+1 \end{cases}$$

3. Numerical results

In the numerical computations it was assumed that the total length of the traverse was 200 km consisting of 20 traverse legs of length 10 km. The accuracy of the end coordinates was set to ± 50 cm or ± 25 cm for satellite Doppler positioning and ± 5 cm for GPS. Furthermore the following standard errors were used in the traversing:

$$\sigma_d = 5 \text{ mm} + 1 \text{ ppm}$$

and

$$\sigma_\beta = 0.3 \text{ mgon}$$

Also the standard error of possible astronomical azimuths were set to 0.3 mgon. By putting very high standard errors at some observations these observations can be regarded as excluded from the observation scheme. The result of the computations are depicted in Figures 2-4.

Figure 2 reveals that a single azimuth observation at the middle of the traverse (P_m) does not pay, while azimuth observations at the end points and the middle drastically reduces the transversal standard error. However, Doppler observations at P_m can very well replace the cumbersome azimuth observations at P_m and the end points. These conclusions are even more pronounced when including GPS observations (Figure 4). Finally the longitudinal standard error σ_y (Figure 3) is

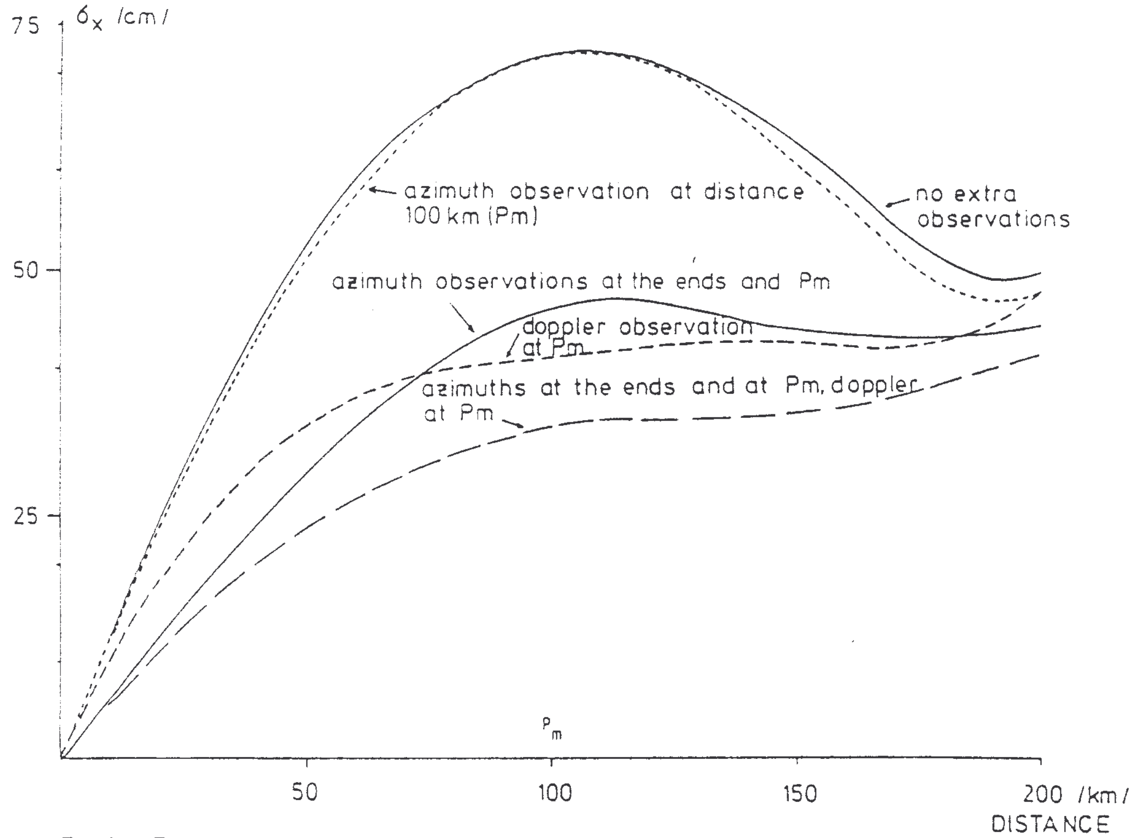

Fig 2. The transversal standard errors in an open traverse. Satellite doppler fixes with coordinate standard error ±50 cm at end points.

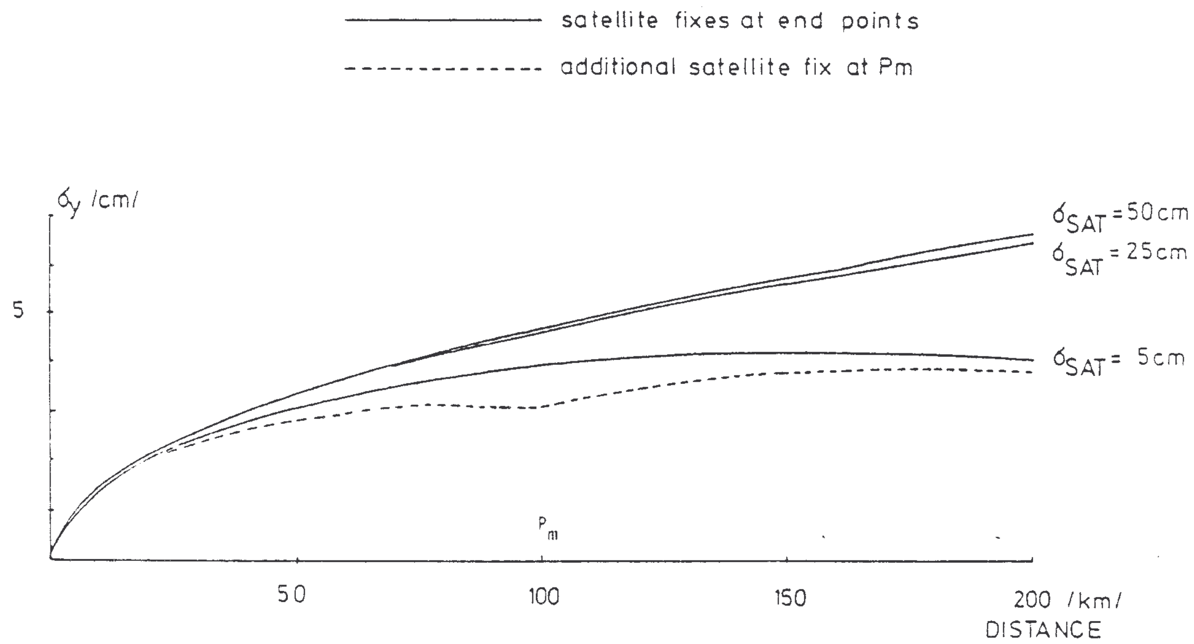

Fig 3. The longitudinal standard error (σ_y) along an open traverse.

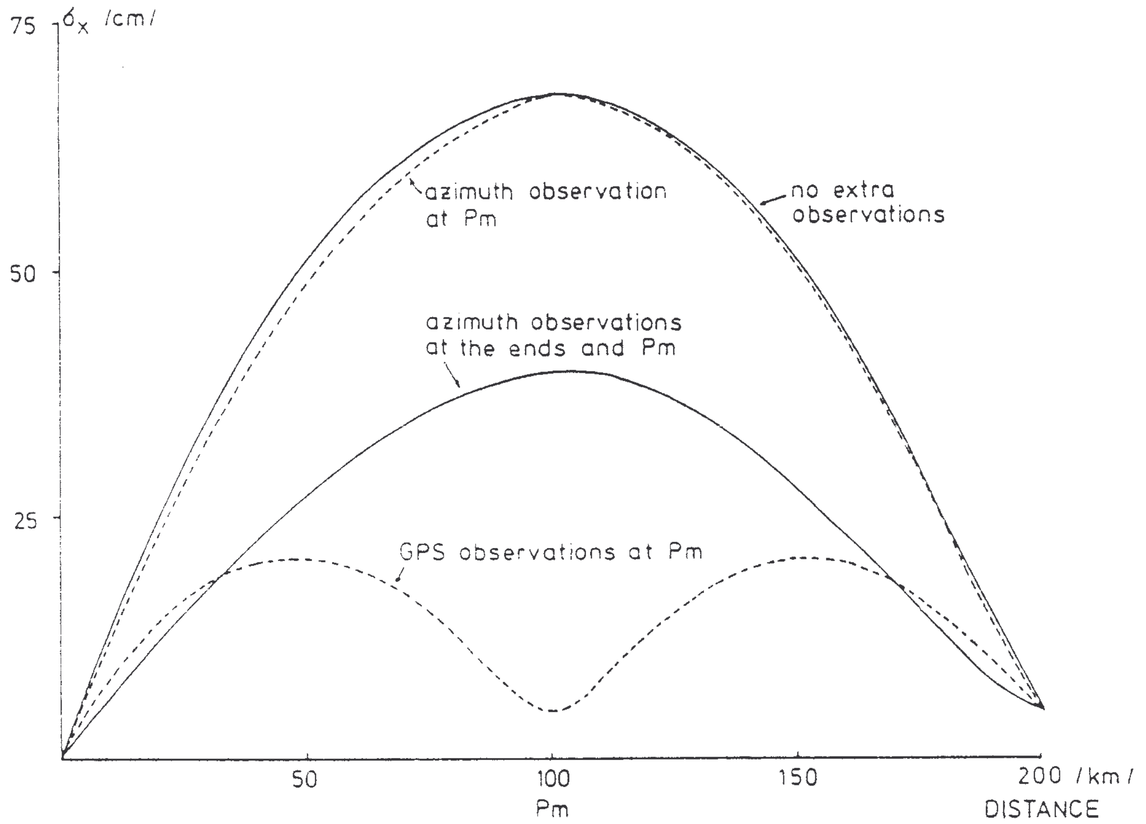

Fig 4. The transversal standard error in an open traverse with GPS fixes at the ends (σ_{SAT}=±5cm).

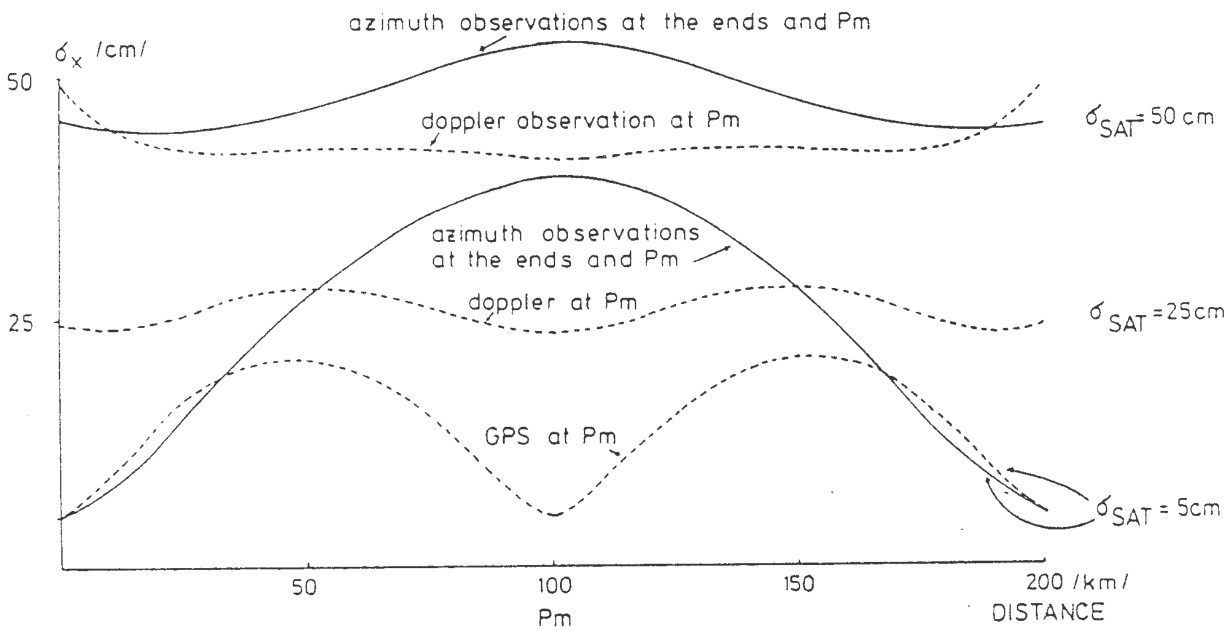

Fig 5. The transversal standard error for bridging with satellite positioning at the end points and possibly at Pm=100km.

not affected by the errors of azimuth or direction observation but is small due to the assumed high accuracy EDM observations.

4. Concluding remarks

The numerical computations have shown that the break-down of a satellite Doppler network does not need the zero-directions and azimuth strengthening on the middle of the traverses, but the azimuth observations are favourably replaced by an extra Doppler observation at the middle of the traverse. In this way we avoid also the possible systematic effects in combining the astronomical observations and the satellite Doppler observations. The advantage of the satellite positioning is obviously even more pronounced when considering the Global Positioning System (GPS). In the future we can expect that the only real need for astronomical observations in connection with national or densification networks is for the absolute orientation with respect to fixed stars or radio sources of the satellite positioning systems on a global basis.

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