Lantmäteriet
Lantmäteriverket - National Land Survey
S - 801 12 GÄVLE SWEDEN

Tekniska skrifter - Professional Papers 1977 / 2

ADJUSTMENT OF LARGE GEODETIC NETWORKS
METHODS AND EXPERIENCES.

by Ilmar Ussisoo

Gävle 1977

Abstract

TRUT, an acronym for "Triangelnäts Utjämning" (Adjustment of Triangulation Networks) is an EDP programme for adjusting horizontal control survey networks, adapted for CDC 6600. Adjustment is by the variation of parameters using observation equations, and can be carried out on the plane or on any chosen reference ellipsoid.

Input data includes parameters and observations. The parameters are (i) coordinates (fixed or preliminary) for all stations, (ii) scale factors for groups of observed lengths, (iii) orientation parameters for groups of Laplace azimuths. The observations consist of (i) direction series, (ii) distances, (iii) Laplace azimuths, with their à priori errors. Feed-in of correlated observations is also possible in groups consisting of not more that ten observations.

Output data includes (i) adjusted parameters with their accuracy relative to fixed parameters and, if required, the relative error between arbitrarily selected pairs of stations; (ii) residuals for all observations.

The number of unknown parameters which can be adjusted during a single run is 330, e g 165 new horizontal stations, if the network includes no other unknown parameters. Where the number of parameters exceeds 330, the network can be broken down into blocks, which can then be connected using Helmert's method for block adjustments, cf section 2. The division of the input data into blocks can also be carried out by TRUT, cf section 9.

In addition it is possible to feed in observation equations (instead of observations). In this case TRUT is applicable to any adjustment using the method of variation of parameters.

1 INTRODUCTION

In 1966, Rikets allmänna kartverk (Geographical Survey Office of Sweden) had developed the first prototype of TRUT. It was a single run adjustment on the plane with a capacity not exceeding 70 new stations. Practical experiences during the past decade have given rise to lively discussions and intimate cooperation between the computing and field departments which have resulted in the present-day TRUT version, the salient features of which are:

(i) a capacity of 330 new parameters per single run

(ii) for larger networks: automatic division into blocks and connection of those blocks

(iii) ability to compute different scale factors or orientation parameters for different groups of distance or Laplace azimuth observations

(iv) extensive checking functions to detect errors in the input data

(v) far-reaching elimination of human errors through built-in consistency checks

The programme is described in details in sections 3-10. Section 2 gives an example of Helmert block-method, sections 11-13 deal with comments and conclusions.

2 HELMERT BLOCK METHOD

An exhaustive description of the method is found in /1/ p 383; it is also mentioned in /2/ p 324. Therefore, we here confine us to a descriptive example.

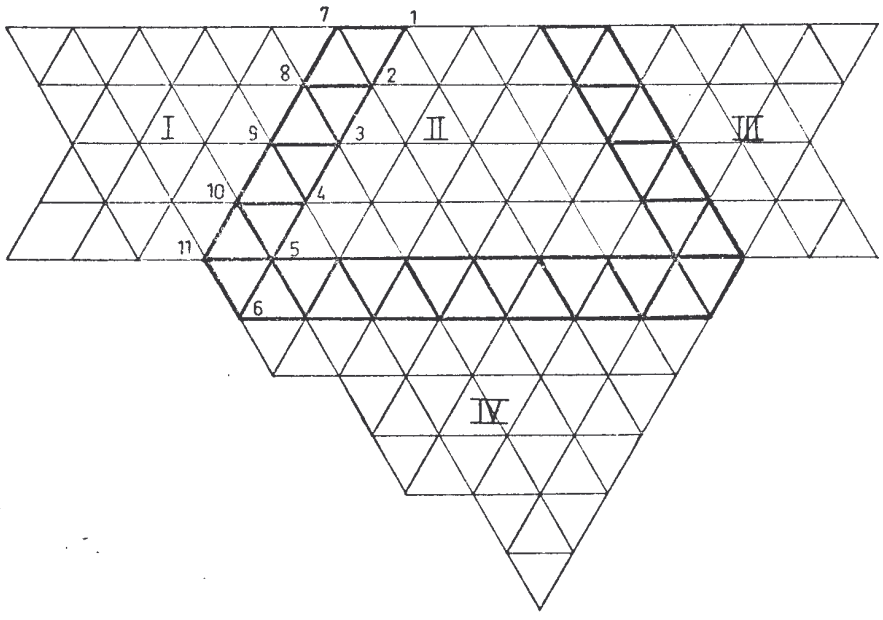


Fig.1

Fig 1 shows a triangulation network which, for adjustment purposes, has been broken down into four blocks. In this particular case, the blocks are bounded by, so called, buffer zones; but division into blocks by boundary lines is also possible. Stations which fall within the buffer zones are referred to below as junction or outer stations whilst the stations within the blocks are called inner stations. Observations which fall within a buffer zone (cf fig 2) can be referred to whichever of the adjacent blocks that is chosen but can, of course, only be included once with full weight.

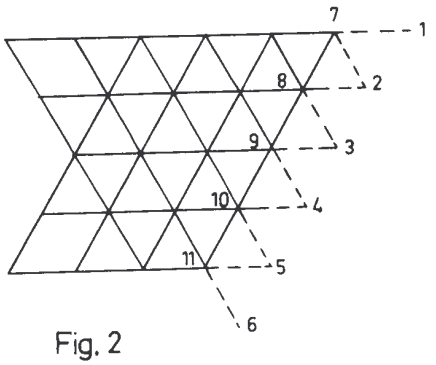


Fig. 2

Fig 2 illustrates how a break-down is carried out in practice in a classical triangulation network. The observations at the stations 7-11 are, in this case, referred to block I.

If the buffer zones were to be replaced by boundary lines the number of junction stations would be reduced by half, but, on the other hand, series of directions observed at the boundary points would belong to both blocks. If, however, the network is a pure trilateration, a breakdown into blocks by boundary lines does not suffer from this disadvantage.

Each element in the normal equations can be considered as belonging to two unknowns. For example, the element which is formed by multiplying column y_{41} with column x_{22} belongs to the unknowns y_{41}/x_{22}. If the x coordinate is denoted with the station number and a plus sign and the y coordinate with the station number and a minus sign the element can be defined by -41/+22. The diagonal element will thus be of the type +i/+i or -i/-i and the right-hand side element $\pm i/h$. The right hand side will be denoted with 900 000. Due to symmetry in the normal equations i e element i/j =element j/i normally only the upper right-hand side half of the equation system is computed and stored by the computer.

Normal equations are formed separately for each block whereby the outer unknowns are placed last. The equations are then reduced so that all the inner unknowns are eliminated (see Fig 3). Thus, for each block a buffer matrix is obtained, cf section 8.

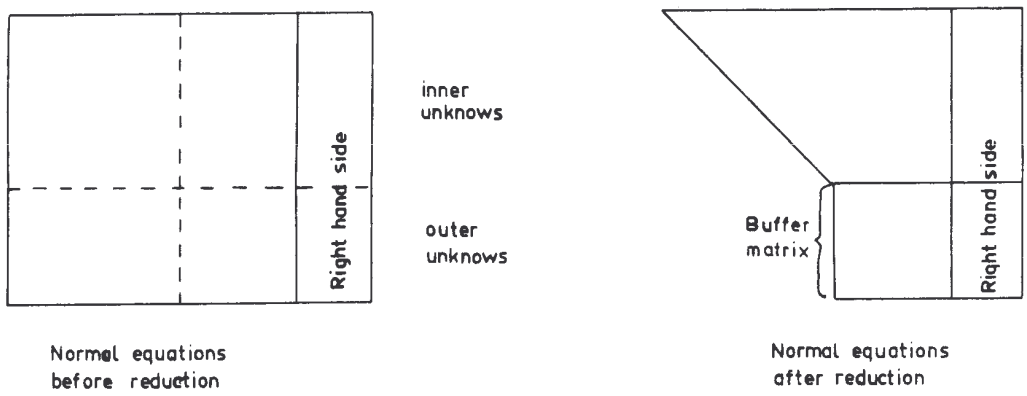


Fig. 3

The next step is to add together all the buffer matrices, term by term, to form a sum matrix. If the accuracy is not the same in all blocks, the buffer matrices are divided by the square of the standard error of the respective blocks before addition. The solution of the sum matrix gives the adjusted values for the outer unknowns.

If the total number of stations in all the buffers exceeds the programme capacity these outer unknowns must, in turn, be broken into inner and outer unknowns and the process described above be repeated. The first stage of the adjustment is said to take place at Level One and the second at Level Two. This process can, theoretically, be continued, level by level, until the number of stations in the sum matrix no longer exceeds the capacity of the computer. However, due to rounding errors the number of levels should be kept within acceptable limits.

Example of the formation of outer matrices

Three blocks with, together, four junction stations are to be put together.

The junction stations are:

Block 1	127, 313, 366
Block 2	127, 313, 478
Block 3	366, 478

σ = the standard errors of unit weight of the blocks which have been determined as 1 000 for Block 1 and Block 2 and 1.414 for Block 3.

Block 1 $L'PL$ = 37.285 Redundancies = 27 σ = 1.000

Unknown nr	+127	-127	+313	-313	+366	-366	900000
+127	+3.178	-0.967	+2.938	-1.917	-2.360	+1.005	+2.403
-127	-0.967	+2.675	+1.317	+0.640	-0.257	-2.438	-2.352
+313	+2.938	+1.317	+3.867	+0.836	-1.322	-3.068	+1.517
-313	-1.917	+0.640	+0.836	+1.996	-1.732	+2.159	+0.013
+366	-2.360	-0.257	-1.322	-1.732	+1.735	+2.336	+0.756
-366	+1.005	-2.438	-3.068	+2.159	+2.336	+2.617	+1.182

Block 2 $L'PL$ = 29.267 Redundancies = 36 σ = 1.000

Unknown nr	+478	-478	+313	-313	+127	-127	900000
+478	+3.675	+2.470	-1.516	-0.937	-3.014	+1.015	+1.336
-478	+2.470	+2.616	+2.941	+3.382	-2.930	-0.664	+2.443
+313	-1.516	+2.941	+3.074	+1.175	-3.587	+2.635	+0.777
-313	-0.937	+3.382	+1.175	+3.675	+1.770	-0.453	-0.532
+127	-3.014	-2.930	-3.587	+1.770	+2.197	+2.122	-2.338
-127	+1.015	-0.664	+2.635	-0.453	+2.122	+2.737	+1.891

Block 3 $L'PL$ = 66.268 Redundancies = 72 σ = 1.414

Unknown nr	+366	-366	+478	-478	900000
+366	+2.866	+3.178	-1.242	-0.634	-0.884
-366	+3.178	+3.426	+2.374	+1.946	+2.442
+478	-1.242	+2.374	+2.854	-3.040	+1.886
-478	-0.634	+1.946	-3.040	+2.678	+1.020

Sum matrix $L'PL$ = 99.686 Redundancies = 135

Unknown nr	+127	-127	+313	-313	+366	-366	+478	-478	900000
+127	+5.375	+1.155	-0.649	-0.147	-2.360	+1.005	-3.014	-2.930	+0.065
-127	+1.155	+4.797	+3.952	+0.187	-0.257	-2.438	+1.015	-0.664	-0.461
+313	-0.649	+3.952	+6.941	+2.011	-1.322	-3.068	-1.516	+2.941	+2.294
-313	-0.147	+0.187	+2.011	+5.671	-1.732	+2.159	-0.937	+3.382	-0.519
+366	-2.360	-0.257	-1.322	-1.732	+3.168	+3.925	-0.621	-0.317	+0.314
-366	+1.005	-2.438	-3.068	+2.159	+3.925	+4.330	+1.187	+0.973	+2.403
+478	-3.014	+1.015	-1.516	-0.937	-0.621	+1.187	+5.102	+0.950	+2.279
-478	-2.930	-0.664	+2.941	+3.382	-0.317	+0.973	+0.950	+3.955	+2.953

Element -127/+313 in the sum matrix is obtained as follows:

from Block 1	1.317 : 1.000
from Block 2	2.635 : 1.000
from Block 3	—
in the sum matrix	1.317 + 2.635 = 3.952

Similarly for element +478/+478:

from Block 1	—
from Block 2	3.675 : 1.000
from Block 3	2.854 : 2.000
in the sum matrix	3.675 + 1.427 = 5.102

Similarly for $L'PL$ = 900 000/900 000

from Block 1	37.285 : 1.000
from Block 2	29.267 : 1.000
from Block 3	66.268 : 2.000
in the sum matrix	37.285 + 29.267 + 33.134 = 99.686

The number of redundancies in the block = total number of observations minus the total number of inner unknowns, or, in the normal case:
total number of observations minus the total number of eliminated station corrections minus twice the total number of inner stations.

The number of redundancies in the sum matrix = the total number of redundancies in the blocks. Thus, in the case above, the number of redundancies in the sum matrix =
27 + 36 + 72 = 135
(Note: the number of redundancies must not be divided by σ^2).

3 INPUT DATA

Computation can be carried out on the plane or on any chosen reference ellipsoid.

The following data are read in:

1 Coordinates of the fixed stations.
2 Approximate coordinates for the inner stations.
3 Approximate coordinates for the junction stations.
4 Values of the fixed scale factors (for groups of observed distances)
5 Approximate values for the inner scale factors.
6 Approximate values for the outer scale factors.
7 Values of the fixed orientation parameters (for groups of observed Laplace azimuths).
8 Approximate values for the inner orientation parameters.
9 Approximate values for the outer orientation parameters.
10 List of pairs of stations for which the relative standard errors are to be computed.
11 Observed series of directions, with à priori standard error.
12 Measured distances without scale factor, with à priori standard error.
13 Astronomic azimuths without orientation parameter, with à priori standard error.
14 Contributions to normal equations.
15 Measured distances with scale factor, with à priori standard error.
16 Astronomic azimuths with orientation parameter, with à priori standard error.

When correlated observations are present, each group of data 12 resp 13 is followed by the weight matrix. For computation on the ellipsoid the major semiaxis and the square of the eccentricity of the reference ellipsoid are also fed in.

The coordinates (data 1-3) are given in metres for computation on the plane, and in centesimal degrees or sexagesimal degrees, minutes and seconds for computation on the ellipsoid. Cf encl 1.

The scale factors (data 4-6) are given with their number and (value minus one) in units of ppm. The orientation parameters (data 7-9) are given with their number and value in centesimal seconds. Cf encl 2.

Direction measurements (data 11) are given in series, in either centesimal degrees or sexagesimal degrees, minutes and seconds. Each direction should have an à priori standard error (cf encl 3). Direction series may contain repeated pointings on the same station (e g R 0 closure). The number of pointings in a series must not exceed twelve. Station adjustments are carried out beforehand only in those cases where the adjusted directions remain uncorrelated, e g by Schreiber´s method or by full rounds. Otherwise all the observed series are fed in unadjusted.

Measured distances are given in millimetres with à priori standard error for each distance, without (data 12) or with (data 15) scale factor. Cf encl 4.

Measured azimuths are given with à priori standard error for each azimuth, without (data 13) or with (data 16) orientation parameter. Cf encl 4.

Should these à priori standard errors for any reason be omitted, the computer automaticallly assumes a value of 1^{cc} for observed directions and azimuths and 1 cm for measured distances. It is also possible to multiply à priori standard errors for a group by a constant.

No reductionc are carried out during run time which means that field data must have been reduced to the centre marks and to the reference ellipsoid or projection plane before being fed into the computer. The feed out from other pre-adjustment programmes is such that it can be fed into TRUT without the need for any changes in form or content.

In the case of astronomic azimuths the observed value A must also be reduced to the geodetic azimuth $\bar{\alpha}_g$ as follows

$$\bar{\alpha}_g = A - (\Lambda - \lambda) \sin \phi$$

where Λ and λ are the astronomic and geodetic longitudes for the Laplace station and ϕ the latitude. Where computation is carried out on the plane the additional correction $(\delta - c)$ is required;

δ = arc to chord correction

c = convergence of the meridian

$\bar{\alpha}_g$ must be computed from final coordinates which at this stage of the computation are unknown. Using approximate coordinates for the reduction, α_g is obtained and is related to $\bar{\alpha}_g$ as follows

$$\bar{\alpha}_g = \alpha_g + d\lambda \sin \phi - (\Lambda - \lambda) \cos \phi \, d\phi \quad \text{(ellipsoid)}$$

or

$$\bar{\alpha}_g = \alpha_g + d\lambda \sin \phi - (\Lambda - \lambda) \cos \phi \, d\phi + d\delta - dc \quad \text{(plane)}$$

The term $(\Lambda - \lambda) \cos\phi \, d\phi$ can be omitted since even where $(\Lambda - \lambda) \cos \phi = 100^{cc}$ and $d\phi = 10^{cc}$ (= 100 m) its value does not exceed 0.002^{cc}. In addition, it is of no importance whether the astronomic or geodetic latitude is used.

When Transverse Mercator is used, the size of the term $d\delta$ is approximately 0.0077^{cc} $(y \, d\Delta x + \Delta x dy)$, and is thus dependent upon the distance from the Central Meridian and the difference in x between the instrument and object stations. Where $\Delta x < 50$ kms, $d\Delta x < 2$ m, $y < 400$ kms and $dy < 10$ m, $d\delta$ will be $< 0.01^{cc}$.

Where $dy < 10$ m the term dc can be simplified to $d\lambda \sin \phi$

Based on the above assumption $\bar{\alpha}_g$ will be on the plane:

$$\bar{\alpha}_g = \alpha_g + d\lambda \sin \phi - 0 + 0 - d\lambda \sin\phi = \alpha_g$$

and on the ellipsoid:

$$\bar{\alpha}_g = \alpha_g + d\lambda \sin \phi$$

See further section 4 d.

4 OBSERVATION EQUATIONS

4 a Notations

The following symbols are used in section 4:

a	major semi-axis of the reference ellipsoid
c	convergence of the meridian
e	first eccentricity of the reference ellipsoid
e'	second eccentricity of the reference ellipsoid
h	(observed value) minus (from approximate coordinate and parameters computed value)
k	(1 + k) is the approximate value of the scale factor
dk	sought increment to k
l	observed value
s	side length computed from approximate coordinates
t	tg ϕ
v	correction to observed value
x, y	approximate plane coordinates
$d\underline{x}$, $d\underline{y}$	sought increments to x, y
$d\bar{x}$, dy	sought increments to ϕ, λ, in units of length
z	approximate value for the bearing or azimuth of the zero of observed directions
dz	sought increment to z
A	observed astronomic azimuth
M	$a(1 - e^2) : W^3$ = radius of curvature in the meridian
N	a : W = radius of curvature in the prime vertical
V^2	$1 + \eta^2$
W^2	$1 - e^2 \sin^2 \phi$
α	azimuth computed from approximate coordinates
α_g	$A - (\Lambda - \lambda) \sin \phi$ = geodetic azimuth derived from astronomic observations
$\bar{\alpha}_g$	α_g computed with adjusted geodetic longitude, i e $(\lambda + d\lambda)$
β	approximate value of the orientation parameter of the Laplace azimuth
$d\beta$	sought increment to β
δ	arc to chord correction
η	$e' \cos \phi$
κ	coefficient of β
Λ	astronomic longitude
λ	approximate geodetic longitude
$d\lambda$	sought increment to λ
ρ	constant for converting from radians
ϕ	approximate geodetic longitude
$d\phi$	sought increment to ϕ
ψ	plane bearing computed from approximate coordinates

The index, o, refers to the instrument station and, i, to the object station. The values of t, M, N, V, W, and η^2 are computed for $\phi = (\phi_o + \phi_i) : 2$.

4 b Observation equations on the plane

ψ and s are computed as follows:

$$\psi = \operatorname{arctg} \frac{y_i - y_o}{x_i - x_o}$$

$$s = \sqrt{(x_i - x_o)^2 + (y_i - y_o)^2}$$

The observation equations become:

for observed directions

$$v = \frac{\rho}{s^2} \{(y_i - y_o)dx_o - (x_i - x_o)dy_o + (y_o - y_i)dx_i - (x_o - x_i)dy_i \} - dz - (l + z - \psi)$$

The term dz is normally eliminated during the build up of normal equations

for distance measurements

$$v = -\frac{1}{s} \{(x_i - x_o)dx_o + (y_i - y_o)dy_o + (x_o - x_i)dx_i + (y_o - y_i)dy_i \} - sdk - (l + kl - s)$$

The equations for data 12 (cf section 3) do not include the terms with k and dk

for azimuth observations

$$v = \frac{\rho}{s^2} \{(y_i - y_o)dx_o - (x_i - x_o)dy_o + (y_o - y_i)dx_i - (x_o - x_i)dy_i \} - \kappa \cdot d\beta - (l + \kappa\beta - \psi)$$

where

$$l = \alpha_g = A_o - (\Lambda_o - \lambda_o) \sin\phi_o - c_o + \delta_o$$

κ = coefficient (e g sin ϕ_o), taken from the data card
When $\kappa = 0$, the programme assumes the value 1.0

The equations for data 13 (cf section 3) do not include the terms with β and $d\beta$.

4 c The Mid-Latitude Formula

On the ellipsoid α and s are computed using the Mid-Latitude Formula

$$P = s\cos\alpha = \frac{M(\phi_i - \phi_o)}{\rho}\left\{1 - \frac{(\lambda_i - \lambda_o)^2}{24\rho^2}(2 + 3t^2 + 2\eta^2)\cos^2\phi - \right.$$

$$\left. - \frac{(\phi_i - \phi_o)^2\,\eta^2}{8\rho^2 V^4}(t^2 - 1 - \eta^2 - 4\eta^2 t^2)\right\}$$

$$Q = s\sin\alpha = \frac{N\cos\phi(\lambda_i - \lambda_o)}{\rho}\left\{1 - \frac{(\lambda_i - \lambda_o)^2\sin^2\phi}{24\rho^2} + \right.$$

$$\left. + \frac{(\phi_i - \phi_o)^2}{24\rho^2 V^4}(1 + \eta^2 - 9\eta^2 t^2)\right\}$$

$$R = \alpha_i - 180^\circ - \alpha_o = (\lambda_i - \lambda_o)\sin\phi\left\{1 + \frac{(\lambda_i - \lambda_o)^2}{12\rho^2}V^2\cos^2\phi + \right.$$

$$\left. + \frac{(\phi_i - \phi_o)^2}{24\rho^2 V^4}(3 + 8\eta^2 + 5\eta^4)\right\}$$

$$s = \sqrt{P^2 + Q^2}$$

$$\alpha_o = \text{arctg}\,\frac{Q}{P} - \frac{R}{2}$$

$$\alpha_i = \text{arctg}\,\frac{Q}{P} + \frac{R}{2} + 180^\circ$$

The formula is correct to within 1 mm for distances less than 175 kms at latitude 47°. At higher latitudes the accuracy decreases and at latitude $69^\circ.5$ the error over a distance of 100 kms is 0.02^{cc} in azimuth and 1.5 mm in distance. Errors increase with the 5-th power of the distance.

4 d Observation equations on the ellipsoid

By substitution

$$\frac{M}{\rho}\, d\phi = d\bar{x} \qquad \frac{N \cos}{\rho}\, d\lambda = d\bar{y}$$

the observation equations become (α_o, α_i and s are computed with the Mid-Latitude formula):

for observed directions:

$$v = \frac{\rho}{s} \{\sin \alpha_o \left(1 - \frac{s^2}{3\,MN}\right) d\bar{x}_o - \cos \alpha_o \left(1 - \frac{s^2}{3\,MN}\right) d\bar{y}_o$$

$$+ \sin \alpha_i \left(1 + \frac{s^2}{6\,MN}\right) d\bar{x}_i - \cos \alpha_i \left(1 + \frac{s^2}{6\,MN}\right) d\bar{y}_i\}$$

$$- (dz - d\lambda_o \sin \phi_o) - (l + z - \alpha_o)$$

The term ($dz - d\lambda_o \sin \phi_o$) is constant for all observations in a series. It is normally eliminated during the formation of normal equations

for distance measurements

$$v = - \cos \alpha_o\, d\bar{x}_o - \sin \alpha_o\, d\bar{y}_o - \cos \alpha_i\, d\bar{x}_i - \sin \alpha_i\, d\bar{y}_i$$

$$- s\, dk - (l + kl - s)$$

Cf corresponding comments in section 4 b

for azimuth observations

$$v = \frac{\rho}{s} \{\sin \alpha_o \left(1 - \frac{s^2}{3\,MN}\right) d\bar{x}_o - \cos \alpha_o \left(1 - \frac{s^2}{3\,MN}\right) d\bar{y}_o$$

$$+ \sin \alpha_i \left(1 + \frac{s^2}{6\,MN}\right) d\bar{x}_i - \cos \alpha_i \left(1 + \frac{s^2}{6\,MN}\right) d\bar{y}_i \}$$

$$+ d\lambda_o \sin \phi_o - \kappa d\beta - (l + \kappa\beta - \alpha_o)$$

Here

$$l = A_o - (\Lambda_o - \lambda_o) \sin \phi_o + d\lambda_o \sin \phi_o$$

and, therefore, cancels out the term $d\lambda_o \sin \phi_o$ in the formula above. Hence the observation equation may be written

$$v = - \frac{\rho}{s} \{\qquad\} - \kappa d\beta - \{A_o - (\Lambda_o - \lambda_o) \sin \phi_o + \kappa\beta - \alpha_o\}$$

On the card with observation data the values for A_o, Λ_o and κ are given. λ_o, ϕ_o and β are obtained from data 1-3° and 7-9°. The computation of the last brace is carried out by the programme. Cf also corresponding comments in section 4 b.

4 e Standardization of the observation equations.

Before forming the normal equations the observation equations must be standardized. This is done by dividing them with the à priori standard error whereby the standardized observation equations become dimensionless. In this connection, the unit used for defining the standard error must be considered. In the examples discussed below the computation is carried out on the plane and without scale factor or orientation parameter, but the reasoning and conclusions can also be applied to the other cases.

The observation equation for an observed direction is

$$v = \rho\left\{\left(\frac{y_i - y_o}{s^2}\right)(dx_o - dx_i) - \left(\frac{x_i - x_o}{s^2}\right)(dy_o - dy_i)\right\} - dz - (l + z - \psi)$$

v is, here, expressed in angular measure which means that even dz and $h = l + z - \psi$ **must be expressed in the same units. Furthermore, the first** brace must be dimensionless and this can most easily be achieved by expressing $(y_1 - y_o)$, $(x_1 - x_o)$ s, dx and dy also in the same length units. **After standardization, the whole observation equation will be dimensionless which means that the à priori standard error for the observed direction must be in the same** angular units as the correction v and the value h.

This reasoning applies also to azimuth observations.

The observation equation for distance measurements is:

$$v = -\frac{x_i - x_o}{s}(dx_o - dx_i) - \frac{y_i - y_o}{s}(dy_o - dy_i) - (l - s)$$

v and $h = l - s$ must be expressed in the same length units, in the **units used for** dx and dy. **According to the reasoning above, the à priori standard error must also be expressed in these units.**

In TRUT, the units are, during the computing stage;

for directions and azimuths	1^{cc}
for distances	**1 decimetre**

At the input and output stages other units are used but these are converted by the computer to and from centesimal seconds and decimetres.

5 FORMATION OF NORMAL EQUATIONS

Here we shortly recapitulate what was stated in section 2, concerning the numeration of the elements in the normal equations. For instance, the element which is formed by multiplying the column belonging to the y-coordinate of the station 10050 with the column belonging to the x-coordinate of 10352 will be defined by - 10050/+10352. The column of the right hand side is denoted with 900000, a scale factor with its number increased with 700000, an orientation parameter with its number increased with 600000.

Each element in the normal equations matrix is allocated a fixed position in the core storage. The position depends upon the sequence in which the approximate coordinates, scale factors and orientation parameters are read into the computer.

The observation equations are not stored during the run time. An observation of uncorrelated distance or azimuth is fed into the computer, its observation equation is formed and standardized, the contributions to elements of normal equations are computed and added to the allocated cells in the core storage. Then the next observation is fed in and the above process is repeated. This process is illustrated by the following example:

The distance between stations 31 and 53 has been measured. The scale factor number is 67. The observation equation becomes

$$v = a\,dx_{31} + b\,dy_{31} + f\,dx_{53} + g\,dy_{53} - \frac{s}{m}\,dk_{67} - \frac{h}{m}$$

$$a = \frac{x_{31} - x_{53}}{ms} = -f \qquad b = \frac{y_{31} - y_{53}}{ms} = -g$$

h = as in section 4 a

$$s = \sqrt{(x_{31} - x_{53})^2 + (y_{31} - y_{53})^2}$$

when the computation is carried out on the plane and when
m = à priori standard error of this observation.

The observation gives rise to the following contributions:

To the element		
	+31/ +31	aa
	+31/ -31	ab
	+31/ +53	af
	+31/ -53	ag
	+31/700067	-as:m
	+31/900000	ah:m
	-31/ -31	bb

-31/ +53	bf
-31/ -53	bg
-31/700067	-bs:m
-31/900000	bh:m
+53/ +53	ff
+53/ -53	fg
+53/700067	-fs:m
+53/900000	fh:m
-53/ -53	gg
-53/700067	-gs:m
-53/900000	gh:m
700067/700067	ss:mm
700067/900000	-sh:mm
900000/900000	hh:mm

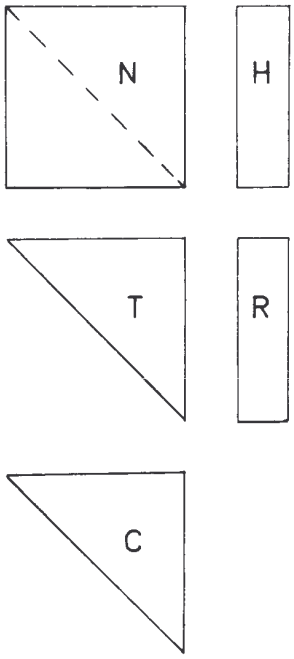
As to the correlated observations of distances or azimuths, the procedure is as follows:

The mutually correlated measurements are given in groups together with their non-diagonal weight matrix P (cf encl 5). From each group the attaching matrices $A'PA$, $A'PL$ and $L'PL$ (where A = matrix of observation equations in the group and L = column matrix with elements h) are built up and their elements are then added to the allocated cells in core storage.

A direction measurement gives rise to an observation equation comprising besides the sought coordinate increments dx, dy also the station correction dz (cf section 4 b) or dz + $d\lambda_o \sin \phi_o$ (cf section 4 d). Each station correction occurs only in one series. Therefore, the directions are given in series, and the attaching matrices $A'PA$, $A'PL$ and $L'PL$ are built up, whereby the station correction is eliminated. This can be done using, for instance, Schreiber's method. In TRUT another method is used the description of which, however, falls outside the scope of this paper. Cf /3/.

One more type of input data is possible: ready-made contributions to normal equations (cf encl 6). It is generally used for reading in the buffer matrices, but it is also useful where the information differs from the before mentioned types. In these cases the contributions to normal equations are, in a suitable way, computed beforehand and fed in with this input-routine for further treatment.

6. SOLUTION OF NORMAL EQUATIONS



$N_{n,n}$ = normal equation matrix = $A'PA$

$n_{i,j}$ = element in the i:th row and j:th column

$H_{n,1}$ = right-hand side = $A'PL$

h_i = element in the i:th row

h_i should not be confused with h^i in the previous sections 4 and 5

Here $h_i = \Sigma(ah:m)$

$\triangleleft T_{n,n}$ = reduced triangular matrix

$R_{n,1}$ = reduced right-hand side

$\triangleleft C_{n,n}$ = inverse triangular matrix = T^{-1}

In Cholesky's elimination method a triangular matrix T is sought such that

$$T'T = N$$

The relationship between the elements in N and T will be, where $i \leq j$

$$t_{ii} = \sqrt{n_{ii} - \sum_{m=1}^{i-1} t_{mi}\, t_{mi}}$$

$$t_{ij} = (n_{ij} - \sum_{m=1}^{i-1} t_{mi}\, t_{mj}) : t_{ii}$$

$$t_{ij} = 0 \quad \text{for} \quad i > j$$

For the right hand side the following applies

$$r_i = (h_i - \sum_{m=1}^{i-1} t_{mi}\, r_m) : t_{ii}$$

Using the above formulae the elements n_{ij} and h_i in the core storage are successively replaced by t_{ij} and r_i respectively. The term $L'PL$ is replaced by

$$L'PL - R'R = V'PV$$

The increments u_i to approximate coordinates are obtained using the formula

$$u_i = (r_i - \sum_{m=i+1}^{n} u_m t_{im}) : t_{ii}$$

and computing backwards from the last unknown u_n. The values u_1 and u_2 are thus the increments to the x and y coordinates respectively, for the first new station.

For computations on the ellipsoid the increments u_x and u_y are transferred to

$$d\phi = \frac{u_x}{M} \cdot \rho \quad \text{and} \quad d\lambda = \frac{u_y}{N \cos\phi} \cdot \rho$$

respectively, where

M = radius of curvature in the meridian

N = radius of curvature in the prime vertical, in the actual stations.

For computation of standard errors the inverse C is required. C is computed by successively replacing the elements in the T matrix as follows:

1. compute all diagonal elements

$$c_{ii} = 1 : t_{ii}$$

2. beginning with the first row, compute row for row from left to right

$$c_{ij} = -\left(\sum_{m=i}^{j-1} c_{im} t_{mj}\right) \cdot c_{jj}$$

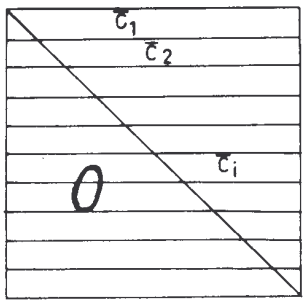
or more simply from a computing point of view

$$c_{ij} = -\left(\sum_{m=i}^{j-1} a_{im} a_{mj}\right) \cdot a_{jj}$$

where a_{ij} are the element values in the mixed matrix T/C.

Also here, the elements t_{ij} are successively replaced by c_{ij}. The normals and their solutions thus require totally $\frac{1}{2} \cdot (n + 1)(n + 2)$ cells in the core storage.

7. COMPUTATION OF STANDARD ERRORS AFTER THE ADJUSTMENT. MODULI CONCEPT



Each row in a matrix can be considered to be a vector where the number of components is equal to the number of columns in the matrix. Cf /4/.

The row vectors in the C matrix will, in the following, be referred to as moduli and the notation for them will be $\bar{c}_1$, $\bar{c}_2$, ..., $\bar{c}_n$.

$$\bar{c}_1 = [c_{11}, c_{12}, c_{13}, \ldots, c_{1n}]$$

$$\bar{c}_2 = [0 \ , c_{22}, c_{23}, \ldots, c_{2n}]$$

- - - - - - - - - - - - - - - - - -

$$\bar{c}_n = [0 \ , 0 \ , 0 \ , \ldots, c_{nn}]$$

where c_{im} = components in modulus $\bar{c}_i$

To each unknown belongs a modulus. Vector algebra can be used by computing with moduli:

1. The scalar product of two moduli

   $\bar{c}_i \cdot \bar{c}_j$

   is defined as $\sum_{m=1}^{n} c_{im} c_{jm}$

2. The absolute value or Euclidean norm of modulus

   $$\left|\bar{c}_i\right| = \left(\sum_{m=1}^{n} c_{im}^2\right)^{\frac{1}{2}}$$

3. The modulus can be considered as a straight line in n-dimensional space. The angle between two moduli $\bar{c}_i$ and $\bar{c}_j$ is

   $$(\bar{c}_i, \bar{c}_j) = \arccos \frac{\bar{c}_i \cdot \bar{c}_j}{\left|\bar{c}_i\right| \left|\bar{c}_j\right|}$$

Moduli can be shown to possess the following properties:

1. The standard error m_i of an unknown, after adjustment, can be determined as

   $$m_i = \pm \sigma \left| \bar{c}_i \right|$$

   with $\sigma = \frac{\text{standard error à posteriori}}{\text{standard error à priori}}$ (= standard error of unit weight)

2. The covariance between two unknowns i and j after adjustment is $\sigma^2 \bar{c}_i \cdot \bar{c}_j$

3. The correlation coefficient between two unknowns i and j after adjustment is equal to the cosine of the angle between the moduli.

4. The law of propagation of error can be written in linear form: If the adjusted quantities are transformed as follows

   $$f = f_x x + f_y y + f_z z$$

   $$g = g_x x + g_y y + g_z z$$

   the moduli for the new quantities f and g will be

   $$\bar{c}_f = f_x \bar{c}_x + f_y \bar{c}_y + f_z \bar{c}_z$$

   $$\bar{c}_g = g_x \bar{c}_x + g_y \bar{c}_y + g_z \bar{c}_z$$

These new moduli can be treated further as described above under 1 - 3. If a new quantity h is formed from f and g ie.

$$h = h_f f + h_g g$$

its modulus will be

$$\bar{c}_h = h_f \bar{c}_f + h_g \bar{c}_g$$

After each completed adjustment the computer is programmed to evaluate the standard error of coordinates for each new station in relation to both the fixed stations (regional standard error)

$$m_{x_i} = \sigma \left| \bar{c}_{x_i} \right|$$

$$m_{y_i} = \sigma \left| \bar{c}_{y_i} \right|$$

$$m_{r_i} = \sqrt{m_{x_i}^2 + m_{y_i}^2}$$

and in relation to those stations, fixed or new, with which it has contact through direct measurement (local standard error). In computing the local standard error, $\bar{c}_{xi}$ och $\bar{c}_{yi}$ are obtained from the submatrix of order two of normal equations which includes only the elements +i/+i, +i/-i, -i/+i, -i/-i.

When required (cf data 10 in section 3) the computer is programmed to evaluate, for selected pairs of stations, also the following

(i) the standard error along the line joining stations, $m_s = \sigma|\bar{c}_s|$

(ii) the standard error perpendicular to the line joining the stations $m_x = \sigma|\bar{c}_t|$

(iii) the relative standard error

$$m_r = \sqrt{m_s^2 + m_t^2} : s$$

The moduli $\bar{c}_s$ and $\bar{c}_t$ are computed as follows:

$$\bar{c}_s = -\{\cos\alpha_i\ \bar{c}_{xi} + \sin\alpha_i\ \bar{c}_{yi} + \cos\alpha_j\ \bar{c}_{xj} + \sin\alpha_j\ \bar{c}_{yj}\}$$

$$\bar{c}_t = \sin\alpha_i\ \bar{c}_{xi} - \cos\alpha_i\ \bar{c}_{yi} + \sin\alpha_j\ \bar{c}_{xj} - \cos\alpha_j\ \bar{c}_{yj}$$

The stations may be chosen in any way, they need not be neighbouring stations. When computing on ellipsoid, however, it must be born in mind that s and α are computed with mid-latitude formula and, therefore, the distance between the stations must not be so large that the formula will give too erroneous results.

For separate computing of the standard error in chosen functions of adjusted coordinates a print-out of the moduli for the final 12 unknowns can be obtained.

8. PROCEDURE FOR CONNECTING BLOCKS

The previous sections have dealt with a completed adjustment. If, however, a block adjustment is to be carried out using Helmert's method the procedures described in Sections 6 and 7 must be modified as follows:

for the outer unknowns the following is obtained in the buffer matrix (= partial adjustment)

$$t_{ij} = n_{ij} - \sum_{m=1}^{k} t_{mi}\, t_{mj}$$

$$r_{ij} = h_{ij} - \sum_{m=1}^{k} t_{mi}\, r_{m}$$

where k is the row number for the last inner unknown.

The term $L'PL$ is reduced by $\sum_{i=m}^{k} r_i\, r_i$

The buffer matrix is punched out on cards as shown on encl 6.

The buffer matrices from the various blocks are then connected as described in Section 2. By the final connection definitive coordinates are obtained for the stations which were included in this stage of the adjustment. Increments to the approximate coordinates for these stations, the standard error of unit weight and the regional standard errors of the coordinates are also obtained. The local standard errors will be erroneous.

If this final connection is carried out at Level One, the easiest way to obtain the adjusted coordinates for inner stations is to readjust all the separate blocks using the previous outer points as fixed points. If, on the other hand, the final connection takes place at a higher level the computation of the inner unknowns in lower level buffers can be computed as described below:

If the unknowns x_i and y_i in the buffer matrix are to take the values dx_i and dy_i computed at a higher level the following contributions must be added

$+i/+i = 10^{12}$ $\qquad$ $+i/900\ 000 = dx_i \cdot 10^{12}$

$-i/-i = 10^{12}$ $\qquad$ $-i/900\ 000 = dy_i \cdot 10^{12}$

The value $V'PV$ will clearly no longer be correct: but since $V'PV$ was obtained from the final connection this is of no importance. The value $V'PV$ must however be positive and, therefore, a contribution

$900\ 000/900\ 000 = \Sigma\ (dx_i\, dx_i + dy_i\, dy_i) \cdot 10^{12}$ is added.

All values obtained for standard errors will be erroneous.

9 DIVISION INTO BLOCKS

The partitioning of the network into blocks was formerly carried out by hand. This was a tedious and time-consuming operation that often also led to erroneous results. The most frequent and critical errors were the following:

(i) a station was entered as outer unknown in one block and as inner unknown in the other

(ii) an outer station had slightly different approximate coordinates in different blocks

In order to eliminate these difficulties the routine BLOCK was developed in 1976 as a preprogramme for TRUT. A detailed description of the programme BLOCK falls outside the scope of this paper and, therefore, only the most important features of the programme are described here.

The boundaries of a block are given as a closed polygon not exceeding ten points. The boundaries need not coincide with the triangulation sides. All stations within this boundary are, to start with, considered as inner stations belonging to the block. This procedure is carried out for all blocks.

Then the observational data is divided between the blocks. As an observation (or a direction series) must not contain stations belonging to different blocks, several inner stations must be altered to outer stations. At the same time also the scale factors and orientation parameters are allocated the right block or labelled as outer unknowns.

As the correct standard errors are obtained only by the final connection process, the stations for which the error investigation is required, must be à priori outer stations. In addition, there might also be stations that must remain as outer stations after the connection process. Therefore, in the input data the stations are divided into four groups:

(i) fixed stations

(ii) stations that may be inner or outer stations

(iii) à priori outer stations

(iv) stations that must remain outer stations after the joining procedure

The scale factors and orientation parameters are divided in the same way.

During the run in BLOCK the whole data deck is divided into suitable blocks, ready to be fed into TRUT.

10 OUTPUT DATA

The output list consists of

(i) the checked input data

(ii) the results of the adjustment

For all observations the value h = (observed value) - (the value computed from approximate coordinates and parameters) is computed. If $|h|$ is larger than the beforehand given tolerance, the observation is rejected and a notation written in the output list.

The checked input data comprise

1 When computation on ellipsoid: the values of the ellipsoid major semi-axis and the square of eccentricity

2 Name, number and (approximate) coordinates for fixed, inner and outer stations

3 Number and (approximate) values of scale factors (fixed, inner, outer)

4 Number and (approximate) values of orientation parameters (fixed, inner, outer)

5 Direction series with their h-values and the z-value for each series

6 Distance measurements with their h-values

7 Azimuth measurements with their h-values

The programme furnishes the following results:

1 Increments (in metres) to the approximate coordinates and their regional and local standard errors.

2 The adjusted values of scale factors with their standard errors

3 The adjusted values of orientation parameters with their standard errors

4 The standard error of unit weight, the number of redundancies and the value $V'PV$

5 The definite coordinates

6 For selected pairs of stations (cf section 7), the values of s, m_s, m_t and m_r

7 Adjusted observations and their corrections

8 When required: standardized observation equations and the normal matrix, triangular matrix and triangular inverse, or only 12 last rows of those matrices.

The following data is punched out:

1 Adjusted values of coordinates

2 For partial adjustments: buffer matrices
For large networks the amount of cards would be too great, and they are, instead, stored on a discfile

11 ADDITIONAL COMMENTS

The most important limitations are

(i) The total number of unknowns (inner and outer) in a block must not exceed 330

(ii) Number of directions in a series must not exceed 12

(iii) For computing on the ellipsoid, the side length must not be longer that 175 kms at latitude 47° and 100 kms at latitude $69^{\circ}.5$, unless the errors in mid-latitude formula exceed 1.5 mm or 0.02^{cc}.

There are no limitations for the amount of observational data.

Some additional features of the programme are the following:

(i) Observations which give rise to h-values exceeding the arbitrary prescribed tolerance, are rejected

(ii) For the connecting process, standardization factors of buffer matrices can be introduced

(iii) Computation of buffer matrices, connecting process and solution (or reduction to a new buffer matrix) of normals can be done during the same run time.

If the number of unknowns in a network does not exceed 330, i e the Helmert Block Method is not applied,

(iv) the programme can be used iteratively: after a first adjustment the approximate coordinates are replaced by the newly adjusted values, and the adjustment is automatically repeated.

Iteration is stopped, when the quotient

$$R'R : L'PL$$

is less than $1 \cdot 10^{-6}$

(v) During the building of normals, a summation of observations to and from each station is carried out. Those new stations, where the number of observations is less than two, are cancelled and the computation is automatically repeated.

It is also possible to feed in observation equations. In this case TRUT is applicable to any adjustment problem. The number of unknowns is then limited to 200.

12 ROUNDING ERRORS

Theoretically, the Helmert Block Method permits the rigorous adjustment of networks, however large they may be. But, in practice, the influence of rounding errors may make the solution invalid.

The loss of computing accuracy can be estimated by the use of the zero-matrix. If a network, containing direction, distance and azimuth observations, is adjusted without any fixed station, but with one outer station, the elements of the buffer matrix (of order 2 x 2) should, theoretically, be noughts, and deviations from a zero value are due to rounding errors.

The satellite traverse Tromsö (Norway) - Catania (Sicily) consists of 373 stations. The total number of observations is 2303. The zero-matrix of Hohenpeissenberg (56 kms SW of Munich) became (units: decimetre and centesimal second):

$$\begin{bmatrix} 0.09 & 0.00 \\ 0.00 & 0.45 \end{bmatrix} \cdot 10^{-6}$$

Diagonal elements of the other unknowns were approximately 0.1 - 1.0. Thus, the computing accuracy can be estimated to 5 - 6 significant digits, i e the influence of rounding errors does not exceed 1 millimetre if the increment does not exceed 100 metres.

13 MISCELLANEOUS EXPERIENCES AND HINTS

TRUT on the ellipsoid has been used mainly for international adjustments, such as RETrig (Re-adjustment of European Triangulation Networks) and satellite traverses. On the plane TRUT has been employed for computation of the new national network of Sweden. The adjusted networks have been of the order 400-800 new stations, and the connection process has generally been carried out at level one. The computation time varies,

depending on the amount of the observational data and the number of unknowns in each block, but as orientation it can be mentioned that an adjustment of a network of 750 new stations, divided into 8 blocks, required 900 system seconds in CDC 6600. This time includes the formation of the buffer matrices, the connection process and the back-solution for the computation of the inner stations. Experiences have shown that for computation of blocks larger than 100 new stations the computation time will increase very rapidly, so the maximum capacity (165 new stations) should be used only if absolutely necessary.

The greatest computation volume, however, is the adjustment of smaller networks on the plane which can be computed in single runs. In order to make the administration easier, and for economical reasons, a reduced version PLANETRUT, with capacity of 100 new stations, has been written out.

TRUT furnishes three kinds of standard errors (cf section 7): regional, local and relative standard errors.

In large networks with only one fixed station the regional standard error is of little value. For example, in the Satellite Traverse Tromsö-Catania the statement "the regional standard error of the station Panker (on the boundary Denmark/ Germany) is 66 cms" means the uncertainty in relation to the single fixed station Hohenpeissenberg in Bavaria, but tells nothing about the accuracy between the stations in the vicinity of Panker. The local standard error might be of greater value, but the best information is obtained from the relative standard errors. As it is hardly possible to compute this quantity for all pairs in the network (n stations give $\frac{n\,(n-1)}{2}$ possible combinations) this investigation is generally carried out for some typical groups of pairs, viz

(i) for pairs of stations located in that part of the network which can be considered as representative for the network as a whole

(ii) for pairs of stations situated in that part of the network where a weaker configuration can give rise to suspicions concerning error accumulation. An example is two adjacent stations without directly connecting measurements

(iii) for pairs of stations located on outer boundaries of the network

(iv) for a number of diagonals across the whole network

In each group the relative standard error for approximately five station pairs should be computed.

The version of TRUT when observation equations are fed in has been applied inter alia to the computation of levelling networks, gravity networks, of astronomical longitudes and to a simultaneous test adjustment of terrestrial and satellite triangulation network.

References

/1/ J J Levallois: Géodésie générale. Tome 2. Paris 1970

/2/ H Wolf: Ausgleichungsrechnung nach der Methode der kleinsten Quadrate. Bonn 1968.

/3/ I Ussisoo: Adjustment of Triangulation Networks. RAK meddelande D 5. Stockholm 1969.

/4/ I Ussisoo: Berechnung mit Fehlervektoren. Allgemeine Vermessungs-Nachrichten 1970, 224-235.

List of coordinates

Station name	No	x or φ	y or λ	
VINGA	10050	6397398.836	1248747.079	on plane [1]
LULEMUS	10352	7568192.801	1783879.676	
VINGA	10050	64.03611701	12.89203718	on ellipsoid centesimal [2]
LULEMUS	10352	75.62806365	25.14464721	
VINGA	10050	573757.01911	113610.20046	on ellipsoid sexagesimal [3]
LULEMUS	10352	680354.92623	223748.65696	

1) x, y are given with the decimal point after the metrevalue. Coordinates may be placed arbitrarily between 32 and 50 (x-coordinate) and 52 and 70 (y-coordinate).

2) If φ, λ are given in centesimal degrees, the decimal point is punched after the degreevalue. Coordinates may be placed arbitrarily between 32 and 50 (latitude) and 52 and 70 (longitude).

3) If φ , λ are given in sexagesimal units, it should be written with degrees in positions 40-41 and 60-61, minutes in 42-43 and 62-63, seconds in 44-51 and 64-71, with decimal point in position 46 and 66 resp.

Encl 1

List of parameters

No	Value	
37	-2·68	Scale factor minus one, in ppm
1693	3·16	
48	-7·7	Orientation parameter, in centesimal seconds
26	1·3	

DIRECTIONS

(1)		(2)		(12)		(21)		
1	Station	215	Directions in centesimal system		S.e à priori cent. sec.			
	Object	(7) 13		(12) 10 07681 9		(21) 10		
	Object	(25) 16		(30) 56 27861 3		(39) 14		
	Object	(43) 18		(48) 113 14625		(57) 18		
	Object	(61) 107		(66) 196 61247		(75) 10	(80)	8

---

(1)		(2)		(12)		(21)		
1		768	Directions in sexagesimal system ° ′ ″					
If a series of directions includes more than four pointings, position 80 is left vacant and the series is continued on the next card. The station number must also be given on this card. The end of a series is indicated by a figure 8 in position 80. A series must not include more than 12 pointings.		(7) 13		(12) 359 57461 9		(21) 10		
		(25) 16		(30) 86 17273 8		(39) 12		
		(43) 219		(48) 128 16173 9		(57) 10		
		(61) 1265		(66) 176 43251 9		(75) 10	(80)	

---

(1)		(2)		(12)		(21)		
1		768						
		(7) 37		(12) 219 14561 9		(21) 18		
		(25) 13		(30) 359 57471 2		(39) 10		
		(43)		(48)		(57)		
		(61)		(66)		(75)	(80)	8

	Station number From	Station number To	Observed value (mm)	S e (mm)		Scale factor number		
2	3768	10352	47678257	61				Data 12
J	168	10050	36857676	49		14		Data 15

	Station number From	Station number To	Observed value (centesimal)	S e cc	Astr longitude ° ′ ″	Orient param number	Coefficient κ	
3	10352	355	47679382 1)	3·24				Data 13 } on plane
M	10050	10048	203764837	2·76		137	1·0	Data 16 } on plane
3	10352	355	51256788 2)	3·24	223736 14			Data 13 } on ellipsoid
M	10050	10048	198236674	2·76	113616 76	137	0·8446	Data 16 } on ellipsoid

1) The observed value is $A - (\Lambda - \lambda) \sin \phi + \delta - c$

2) The observed value is A

Encl 4

Correlated observations

①									
5									2
2	2729	2731	28676587	10					
2	2729	2928	23195368	10					
2	2731	2928	34721786	10	②				
2	2731	2931	18561584	10					
2	2928	2931	29065914	10					
③	0·435	0·075	0·390	-0·220	-0·125	0·445	0·105	0·075	
	-0·155	0·500	0·125	0·090	-0·170	0·115	0·375		

① Number of observations in the group (must not exceed ten)

② Distance measurements, cf encl 4

③ Weight matrix elements: p_{11}, p_{21}, p_{22}, p_{31}, p_{32}, p_{33}, p_{41}, p_{42}, p_{43}, p_{44}, p_{51}, p_{52}, p_{53}, p_{54}, p_{55}.

Each element is punched with the decimal point in arbitrary positions within the allocated field.

Encl 5

	Unknown no	Unknown no	Contribution		Unknown no	Unknown no	Contribution		Name of area
6	-478	1327	-0·257269	E-02	-478	700037	0·176859	E-03	SWEDE
6	-478	600025	-47·5697		-478	900000	43·86768		SWEDE
6	900000	900000	0·57768973	E 02	999999	42			SWEDE

- 478	unknown y_{478}
1327	unknown x_{1327}
700037	Scale factor no 37
600025	orientation parameter no 25
900000	h
999999	number of redundancies (here 42) indication

The value of contribution is given with the decimal point (arbitrary position in the allocated field) or with a 10-power, whereby the letter E is punched in position 32 or 69.
Name of area appears in the output list.

Encl 6