

Geodesi

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## Gauss Conformal Projection (Transverse Mercator)

### Krüger's Formulas

#### Symbols and Definitions

$a$	semi-major axis of the ellipsoid
$f$	flattening of the ellipsoid
$e^2$	first eccentricity squared
$\varphi$	geodetic latitude, positive north
$\lambda$	geodetic longitude, positive east
$x$	grid coordinate, positive north
$y$	grid coordinate, positive east
$\lambda_0$	longitude of the central meridian
$k_0$	scale factor along the central meridian
$\delta\lambda$	difference $\lambda - \lambda_0$
$FN$	false northing
$FE$	false easting

All angles are expressed in radians. Please note that the x-axis is directed to the north and the y-axis to the east.

The following variables are defined out of the ellipsoidal parameters  $a$  and  $f$ :

$$e^2 = f(2 - f)$$

$$n = \frac{f}{(2 - f)}$$

$$\hat{a} = \frac{a}{(1 + n)} \left( 1 + \frac{1}{4}n^2 + \frac{1}{64}n^4 + \dots \right)$$

## Conversion from geodetic coordinates $(\varphi, \lambda)$ to grid coordinates $(x, y)$ .

Compute the conformal<sup>1</sup> latitude  $\varphi^*$

$$\varphi^* = \varphi - \sin \varphi \cos \varphi (A + B \sin^2 \varphi + C \sin^4 \varphi + D \sin^6 \varphi + \dots)$$

The coefficients  $A$ ,  $B$ ,  $C$ , and  $D$  are computed using the following formulas:

$$A = e^2$$

$$B = \frac{1}{6}(5e^4 - e^6)$$

$$C = \frac{1}{120}(104e^6 - 45e^8 + \dots)$$

$$D = \frac{1}{1260}(1237e^8 + \dots)$$

Let  $\delta\lambda = \lambda - \lambda_0$  and

$$\xi' = \arctan(\tan \varphi^* / \cos \delta\lambda)$$

$$\eta' = \operatorname{arctanh}(\cos \varphi^* \sin \delta\lambda)$$

then

$$x = k_0 \hat{a} \left( \begin{array}{l} \xi' + \beta_1 \sin 2\xi' \cosh 2\eta' + \beta_2 \sin 4\xi' \cosh 4\eta' + \beta_3 \sin 6\xi' \cosh 6\eta' + \\ + \beta_4 \sin 8\xi' \cosh 8\eta' + \dots \end{array} \right) + FN$$

$$y = k_0 \hat{a} \left( \begin{array}{l} \eta' + \beta_1 \cos 2\xi' \sinh 2\eta' + \beta_2 \cos 4\xi' \sinh 4\eta' + \beta_3 \cos 6\xi' \sinh 6\eta' + \\ + \beta_4 \cos 8\xi' \sinh 8\eta' + \dots \end{array} \right) + FE$$

<sup>1</sup> Older Swedish literature refers to this quantity as the isometric latitude. Today the term isometric latitude is applied to the quantity

$\psi = \ln \{ \tan(\pi/4 + \varphi/2) [(1 - e \sin \varphi)/(1 + e \sin \varphi)]^{e/2} \}$ . The isometric latitude is related to the conformal latitude by  $\psi = \ln \tan(\pi/4 + \varphi^*/2)$ . Cf. John P. Snyder: Map Projections - A Working Manual, U.S. Geological Survey Professional Paper 1395.

where the coefficients  $\beta_1, \beta_2, \beta_3$  and  $\beta_4$  are computed by

$$\beta_1 = \frac{1}{2}n - \frac{2}{3}n^2 + \frac{5}{16}n^3 + \frac{41}{180}n^4 + \dots$$

$$\beta_2 = \frac{13}{48}n^2 - \frac{3}{5}n^3 + \frac{557}{1440}n^4 + \dots$$

$$\beta_3 = \frac{61}{240}n^3 - \frac{103}{140}n^4 + \dots$$

$$\beta_4 = \frac{49561}{161280}n^4 + \dots$$

### Conversion from grid coordinates $(x,y)$ to geodetic coordinates $(\varphi,\lambda)$

Introduce the variables  $\xi$  and  $\eta$  as

$$\xi = \frac{x - FN}{k_0 \cdot \hat{a}}$$

$$\eta = \frac{y - FE}{k_0 \cdot \hat{a}}$$

Let

$$\xi' = \xi - \delta_1 \sin 2\xi \cosh 2\eta - \delta_2 \sin 4\xi \cosh 4\eta - \delta_3 \sin 6\xi \cosh 6\eta - \delta_4 \sin 8\xi \cosh 8\eta - \dots$$

$$\eta' = \eta - \delta_1 \cos 2\xi \sinh 2\eta - \delta_2 \cos 4\xi \sinh 4\eta - \delta_3 \cos 6\xi \sinh 6\eta - \delta_4 \cos 8\xi \sinh 8\eta - \dots$$

where

$$\delta_1 = \frac{1}{2}n - \frac{2}{3}n^2 + \frac{37}{96}n^3 - \frac{1}{360}n^4 + \dots$$

$$\delta_2 = \frac{1}{48}n^2 + \frac{1}{15}n^3 - \frac{437}{1440}n^4 + \dots$$

$$\delta_3 = \frac{17}{480}n^3 - \frac{37}{840}n^4 + \dots$$

$$\delta_4 = \frac{4397}{161280}n^4 + \dots$$

The conformal latitude  $\varphi^*$  and the difference in longitude  $\delta\lambda$  are obtained by the formulas

$$\varphi^* = \arcsin(\sin \xi' / \cosh \eta')$$

$$\delta\lambda = \arctan(\sinh \eta' / \cos \xi')$$

Finally, the latitude  $\varphi$  and the longitude  $\lambda$  are obtained by the formulas

$$\lambda = \lambda_0 + \delta\lambda$$

$$\varphi = \varphi^* + \sin \varphi^* \cos \varphi^* (A^* + B^* \sin^2 \varphi^* + C^* \sin^4 \varphi^* + D^* \sin^6 \varphi^* + \dots)$$

where

$$A^* = (e^2 + e^4 + e^6 + e^8 + \dots)$$

$$B^* = -\frac{1}{6}(7e^4 + 17e^6 + 30e^8 + \dots)$$

$$C^* = \frac{1}{120}(224e^6 + 889e^8 + \dots)$$

$$D^* = -\frac{1}{1260}(4279e^8 + \dots)$$

### Worked example:

ELLIPSOID GRS 1980

Semi-major axis (a)	6378137.0000 m.
Flattening (f)	1/298.257222101

TRANSVERSE MERCATOR PARAMETERS

Longitude of the central meridian	13 35 7.692000	degr. min. sec.
Scale factor on the central meridian	1.000002540000	
False northing	-6226307.8640	m.
False easting	84182.8790	m.

Latitude and longitude	66 0 0.0000	24 0 0.0000	degr. min. sec.
Grid coordinates	1135809.413803	555304.016555	m